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ABSTRACT

This document outlines part one of the third year course of the newly implemented New York State Mathematics Curriculum. The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing the formerly separate topics found in algebra, geometry, and intermediate algebra/trigonometry, and introducing timely topics which have been absent from the course of study. The guide is based upon the scope and sequence for Sequential Mathematics, Course III, part 1. It consists of plans for lessons, which should be considered suggestions rather than prescriptions. There is an abundance of application problems, provided as a resource for teachers. The activities emphasize problem solving skills, reasoning skills, and critical thinking. Though Course III is written with 11th grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less or more than 1 year. (PK)

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Pilot Edition — 1986

SEQUENTIAL MATHEMATICS

Course III

Part I

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FOREWORD

The Sequential Mathematics Sequence provides New York City students with an integrated course of study for grades nine, ten, and eleven. In a departure from traditional courses of study, lessons interweave algebra, geometry, logic, probability, and mathematical systems. This course extends the integrated approach that begins in kindergarten and continues through grade 8.

In the Sequential Mathematics Sequence, algebra, geometry, and trigonometry remain important components of the curriculum. In addition, other branches of mathematics -- such as probability, logic, and mathematical systems -- are introduced and emphasized as important tools for dealing with present and future technological developments.

Each lesson in Course III begins with a clearly stated aim presented in question form. Performance objectives follow; these provide a specific focus for the teacher. The new vocabulary for the lesson is presented. This is followed by a motivational activity. A full development is presented, with suggested pivotal questions. Applications, solutions to applications, and a summary are also included. In each lesson opportunities have been infused for helping students to develop critical thinking, problem-solving, and reasoning skills. These activities will enable students to become more skilled in higher levels of thinking.

The Sequential Mathematics Sequence has been designed to anticipate students' needs for mathematical skills in an increasingly technologically-oriented culture. We anticipate that the Sequential Mathematics Sequence will serve the individual needs of your students and enable them to acquire the mathematics skills they need in the present and for the future.

Charlotte Frank

Charlotte Frank
Executive Director
Division of Curriculum and Instruction

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INTRODUCTION

Sequential Mathematics (Course III, Part I) is the third year course of the newly implemented New York State curriculum. This publication from the Division of Curriculum and Instruction is a guide for teachers. It attempts to present the material covered in this course in a way that is most comprehensive and useful to the classroom teacher.

Traditionally, in New York State, mathematics in grades nine, ten, and eleven has consisted of algebra, geometry, and intermediate algebra/trigonometry. Students completing the three-year sequence have been exposed to these topics as relatively independent units. Moreover, students discontinuing their study of mathematics after grade eleven have never been exposed to such timely and practical topics as probability, statistics, and formal logical reasoning. Those terminating their high school mathematics program after grade ten have been given even fewer opportunities for exposure to these topics. The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing these formerly separate topics and introducing those which have been absent from the course of study.

This curriculum guide is based upon the Scope and Sequence for Sequential Mathematics, Course III, Part I. It consists of plans for lessons, which should be considered suggestions rather than prescriptions. Each of these plans stress the development of mathematical concepts, with an emphasis on providing opportunities for the development of problem solving skills, reasoning skills and critical thinking. To help students develop these skills, each plan contains suggested questions to encourage student investigation and thought. The problems presented in the plans are designed to be relevant to students' experiences. There is an abundance of application problems.
These have been provided as a resource for teachers so that they may select appropriate problems for different levels of classes. Some objectives and application problems are marked with an asterisk and may be more appropriate for above-average classes.

Though Course III is written with eleventh grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less than or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of material to be stressed, or extra material to be presented.

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LESSON: 1

AIM: How can we extend the number system?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State that every point on the real number line can be put into a 1-1 correspondence with a real number.
2. Use the real number line to classify numbers into subsets - natural, whole numbers, integers, and rational.
3. Change a rational number into a repeating decimal and conversely.
4. State that an irrational number cannot be expressed as an infinite repeating decimal.

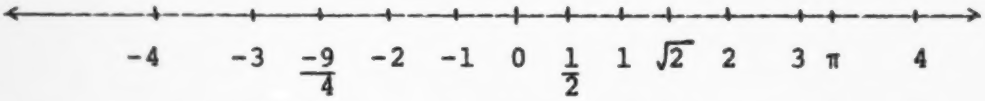
VOCABULARY: real number line, real numbers, natural numbers, whole numbers, integers, rational, repeating decimal, irrational

MOTIVATION:

Draw a real number line. Include natural numbers, a whole number that is not a natural number, integers that are not whole numbers, rational numbers that are not integers, and two irrational numbers.

DEVELOPMENT:

1. An example of a possible answer to the motivation problem is:



The natural numbers on the real number line that were drawn are: 1,2,3,4. The whole number that is not a natural number is 0, and the integers that are not whole numbers are: -4,-3,-2,-1. The rational numbers that are not integers are: $\frac{-9}{4}$, $\frac{1}{2}$. The irrational numbers are: $\sqrt{2}$, π.

2. What are the following subsets of the real numbers: natural numbers, whole numbers, integers, rationals, and irrationals? Elicit: The natural, or counting, numbers are: $N = \{1,2,3,4,\ldots\}$. The whole numbers are: $W = N \cup \{0\}$ or $W = \{0,1,2,3,4,\ldots\}$. The integers are: $Z = W \cup \{-X : X \in W\}$ or $Z = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$. The rational numbers are: $Q = \left\{\frac{a}{b} : a,b \in Z \text{ and } b \neq 0\right\}$. Irrational numbers can not be expressed in the form of $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The set R, of reals, consists of all rational and irrational numbers. Note that there is a one-to-one correspondence between the points on the real number line and the real numbers, that is, for every point on the real number line there is one and only one real number and for every real number there is one and only one point.

3. What are the differences in decimal representation between rational and irrational numbers? Consider examples such as $\frac{1}{2}$, $\frac{1}{3}$, 2, and π. Note that $\frac{1}{2} = .5$, $\frac{1}{3} = .333\ldots$ or $.\overline{3}$, $2 = 1.4142\ldots$, $\pi \approx 3.1415\ldots$ Rational numbers can be expressed as terminating (.5) or repeating decimals ($.\overline{3}$). Irrational numbers can only be expressed as non-terminating, non-repeating decimals. The digit in the next decimal place can not be predicated from any pattern of the previous digits. For example, the next two digits in the decimal representation of π are 92.

4. How do you express the following rational numbers as repeating decimals? $\frac{1}{7}$, $3\frac{5}{8}$. Divide the numerator by the denominator to find the decimal representation of the rational number $\frac{1}{7} = .142857142\ldots$ or $\overline{.142857}$. $3\frac{2}{11} = 3.1818\ldots$ or $3.\overline{18}$.

5. How do you express the following repeating decimal as a fraction? .714714... Let n = .714714... Since there is a 3 digit repetition, multiply n by 1000 to get 1000n = 714.714714... Subtract n from 1000n:

$$\begin{array}{r} n = .714714\ldots \\ \hline 999n = 714 \\ n = \frac{714}{999} \text{ or } n = \frac{238}{333} \end{array}$$

Express .0125125... as a fraction. Let $n = .0\overline{125}$. Since there is one digit followed by a repetition of 3 digits, multiply n by 10000. $10000n = 125.125125...$ To align the repeating digits multiply n by 10. Subtract 10n from 10000n.

$$\begin{array}{r l} 10000n = & 125.125125... \\ 10n = & .125125... \\ \hline 9990n = & 125 \\ n = & \frac{125}{9990} \text{ or } n = \frac{25}{1998}. \end{array}$$

APPLICATIONS:

1. Is .999... equal to 1? Justify your answer.

* 2. For what values of x is $\frac{7x}{x+1}$ equal to:
   a) An integer?
   b) A rational number?

3. Which of the sets R,Q,Z,W,N contain the following numbers:
   a) $-3.\overline{7}$
   b) $\sqrt{41}$
   c) 0

4. Find $\sqrt{2}$ on the number line by construction.

5. If x is a real number greater than or equal to zero, under what conditions is the $\sqrt{x}$:
   a) a rational number;
   b) an irrational number?

* 6. Assume you invest \$1 at 100% interest, compounded every minute. At the end of one year, how much money do you have? To find the amount (A) of money use: $A = P(1+\frac{i}{m})^{nm}$, where P = principal, i = interest rate, m = number of times compounded per year, n = number of years. The answer should approximate the number e. Investigate the applications of e.

* 7. What is the positive solution to the proportion: $\frac{x}{1} = \frac{1}{x+1}$? The number represented by the positive solution is called the golden mean or golden ratio. Investigate its' applications.

SOLUTIONS TO APPLICATION PROBLEMS:

1. Yes. Let

$$\begin{array}{r} 10n = 9.999\ldots \\ \underline{n = .999\ldots} \\ 9n = 9\ ;\ n = 1 \end{array}$$

2. a) Let $\frac{7x}{x+1} = n$, where n is an integer.

   $7x = n(x+1) \quad 7x = nx+n \quad 7x-nx = n$

   $x(7-n) = n \quad x = \frac{n}{7-n}$. All x of the form $\frac{n}{7-n}$,

   where $n \in Z$, $n \neq 7$.

   b) All rational x, except $x = -1$.

3. (a) R,Q

   (b) R,Q,Z,W,N

   (c) R,Q,Z,W

4. Construct a right triangle with legs of 1 unit. The hypotenuse has length $\sqrt{2}$. Copy the length of the hypotenuse on the number line.

5. a) When x is a perfect square of a rational number.

   b) When x is not the perfect square of a rational number.

*6. $A = 1\left(1+\frac{1}{525600}\right)^{1.525600} = 2.718$. This number is approximately equal to e, which is an irrational number. In calculus, the number e has applications for growth and decay of populations.

*7. $x^2+x = 1 \quad x^2+x-1 = 0$. The positive solution is: $x = \frac{-1+\sqrt{5}}{2}$. The golden mean is approximately .618. It has applications to architecture and to a well known series called the Fibonacci series (1,1,2,3,5,8,13,21,...), where each term equals the sum of the previous two terms. The ratio of the nth term to the (n+1)st term approaches the golden mean.

SUMMARY:

1. What are the different subsets of the reals?
2. What is the difference between the decimal representations of rationals and irrationals?
3. How do you convert a repeating decimal to a rational number?

LESSON: 2

AIM: What are some of the properties of real numbers?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Review the definitions of group and field.
2. Illustrate the field properties of the real numbers.
3. State the multiplicative properties of 0 and -1.
4. State and apply the order of operations.

VOCABULARY: closure, associative, identity, inverse, commutative, distributive, operation, group, field

MOTIVATION:

Using real numbers a, b, and c, illustrate the closure, associative, identity, inverse, and commutative properties of addition.

DEVELOPMENT:

1. The properties of addition are:
   a) Closure: $a+b \in R$.
   b) Associative: $a+(b+c) = (a+b)+c$.
   c) Identity: $0 \in R$ is a unique element such that $a+0 = 0+a = a$ for any $a \in R$.
   d) Inverse: For any $a \in R$ there is a unique additive inverse, $-a$, such that $a+(-a) = (-a)+a = 0$.
   e) Commutative: $a+b = b+a$.

   A nonempty set, G, with a binary operation, +, that satisfies the above properties (a) through (d) is a mathematical system called a group. If the group also satisfies property (e), then the group is commutative or abelian.

2. Using real numbers a, b, and c, illustrate the closure, associative, identity, inverse, and commutative properties of multiplication.
   a) Closure: $ab \in R$.
   b) Associative: $a(bc) = (ab)c$.

c) Identity: $1 \in R$ is a unique element such that $a \cdot 1 = 1 \cdot a = a$ for any $a \in R$.

d) Inverse: For $a (a \neq 0)$ in R there is a unique multiplicative inverse, $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

e) Commutative: $ab = ba$.

The distributive property combines addition and multiplication: $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$.

3. Review with the class that a mathematical system is a field if it is a set, F, with two binary operations, + and ·, such that:

   a) $(F,+)$ is a commutative group;

   b) $(F/(0),\cdot)$ is a commutative group ($F/\{0\}$) means set F without the additive identity);

   c) the distributive property holds. An example of a field is the reals $(R,+,x)$. The properties of a field can be used to perform computations and to solve equations. For example, name the properties illustrated for the field of reals:

   d) $\sqrt{3}\,(\sqrt{2} - \sqrt{3}) = \sqrt{6} - 9$ (distributive);

   e) $(19 \cdot \frac{1}{2}) \cdot 6 = 19 \cdot (\frac{1}{2} \cdot 6)$ (associative)

   $= 19 \cdot 3$

   $= 57;$

   f) $x+12 = 15$

   $x+12+(-12) = 15+(-12)$ (inverse)

   $x+0 = 3$ (identity)

   $x = 3$

4. To simplify algebraic and numerical expressions, additional properties of the reals and the order of operations should be reviewed. What is the:

   a) Multiplicative property of 0? $a \cdot 0 = 0 \cdot a = 0$ for any $a \in R$.

   b) Multiplicative property of -1? $a(-1) = (-1)a = -a$ for any $a \in R$.

   c) Order of operations?

      (i) Operations within parentheses (P).

      (ii) Simplify expressions with exponents (E).

      (iii) Perform all multiplications (M) and divisions (D), from left to right.

(iv) Perform all additions (A) and subtractions (S), from left to right.

A mneumonic device to remember the order of operations is: Please Excuse My Dear Aunt Sally.

5. Simplify:
   a) $(5+4^2)\cdot 3-12$. $(5+16)\cdot 3-12 = 21\cdot 3-12 = 63-12 = 51$
   b) $\frac{7-2\cdot 5}{6}$ The numerator is treated as if it were within parentheses.
   $\frac{7-10}{6} = \frac{-3}{6} = \frac{-1}{2}$.
   c) When $a = -1$, $b = -5$, $c = 2$, evaluate $a^3-7bc+4b^2$.
   $$\begin{aligned}(-1)^3-7(-5)(2)+4(-5)^2 &= -1+70+4\cdot 25\\ &= -1+70+100\\ &= 169\end{aligned}$$

APPLICATIONS:

1. When $a = -3$, $b = -4$, $c = 6$, evaluate $\frac{a^2-2ac+b^2}{ab-bc+ac}$

2. State the property of the reals that justifies each step:
$$\begin{aligned}\left(\frac{1}{4}x\right)4 &= 4\left(\frac{1}{4}x\right)\\ &= \left(4\cdot\frac{1}{4}\right)x\\ &= 1\cdot x\\ &= x\end{aligned}$$

3. Simplify and state the property of the reals that justifies each step:
$\frac{1}{5}(5x+5)-x$

4. Find the multiplicative inverse of $.\overline{25}$.

5. Which of the following systems is a group? If the system is not a group, name the group properties that fail.
   a) (Whole numbers, +)
   b) (Integers, ·)
   c) (Rational numbers, +)
   d) (Even integers, +)
   e) (Odd integers, +)
   f) (Integers/$\{0\}$, ·)

* 6. Consider the set V, with elements a,b,c,d and an operation, *. The following table illustrates the operation:

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

a) What is the identity element?

b) What are the inverses of a,b,c, and d?

7. Which of the following systems is a field? If the system is not a field, name the field properties that fail.

a) (Integers, +, ·)

b) (Rational numbers, +, ·)

SOLUTIONS TO APPLICATION PROBLEMS:

1. $$\frac{(-3)^2-2\cdot(-3)(6)+(-4)^2}{(-3)(-4)-(-4)(6)+(-3)(6)} = \frac{9+36+16}{12+24-18} = \frac{61}{18}$$

2. commutative
associative
inverse
identity

3. $\frac{1}{5}.5x+\frac{1}{5}.5-x$ distributive
$= 1\cdot x+1-x$ inverse
$= x+1-x$ identity
$1+x-x$ commutative
$= 1+0$ inverse
$= 1$ identity

4. Convert $.\overline{25}$ to a fraction.

$$\begin{array}{rl} 100n = & 25.2525\ldots \\ \underline{\quad n =} & \underline{\ .2525\ldots} \\ 99n = & 25 \end{array}$$

$99n = 25$; $n = \frac{25}{99}$; Multiplicative inverse is $\frac{99}{25}$.

5. a) no inverses
   b) no inverses
   c) yes
   d) yes
   e) no identity
   f) no inverses

*6. a) "a" is the identity element.
   b) Each element is its own inverse. This group is called the Klein 4 group. It has applications to the psychologist Jean Piaget's formal operations stage.

7. a) no multiplicative inverses
   b) yes

SUMMARY:

1. What is a group?
2. What is a field?
3. What is the order of operations?
4. What are examples of groups?
5. What are examples of fields?

LESSON: 3

AIM: How do we solve first-degree equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Review the meaning of open sentences, conditional and identical equations, domain, and solution set.
2. State the postulates of equality.
3. Use the postulates of equality to solve first-degree equations.

VOCABULARY: open sentence, statement, first degree equation, conditional and identical equations, domain, solution set

MOTIVATION:

Which of the following examples are open sentences:

1. $x-2 = 9$;
2. $x+8 > 12$;
3. $5+4 > 8$;
4. $\frac{1}{3}\cdot 12 = 3$;
5. The mayor of New York City is X;
6. The mayor of New York City is Moynihan?

DEVELOPMENT:

1. Elicit: An open sentence must contain a variable. For the motivation problem, examples 1, 2, and 5 are open sentences. The truth or falsity of the open sentence is not determined until the variable is replaced with elements of the domain or replacement set. Then the open sentence is called a statement, or a closed sentence. There are no variables in a statement. Which of the examples from the motivation problem are statements? Examples 3, 4, and 6 are statements (3 and 4 are true statements, while 6 is a false statement).

2. What is the name of the subset of the domain that consists of those elements of the domain that make the open sentence true? Elicit: It is the solution set. A solution to an equation is often called a root of the equation. We will review how to solve first degree (that is, the exponent of X is 1) equations. These equations are also called linear equations. The solution sets to these equations may be empty, finite, or infinite. Which of the following examples has a solution set which is empty, finite, or infinite?:

   a) $4(x+2) = 4x+8$ Domain is set of reals.
   b) $x+\frac{1}{3} = 5$ Domain is set of integers.
   c) $x+6 = 10$ Domain is set of integers.

   Elicit: The solution set for (a) is the set of reals. Since the equation is true for all replacement values in the domain, the equation is called an identity. The solution set for (b) is the empty, or null set, $\emptyset$. The solution set for (c) is the finite set, (4). Explain to the class that since the equations for (b) and (c) have solution sets which do not contain all possible replacements of the variable, these equations are called conditional equations.

3. Have the students recall that to solve first degree equations, we use properties of equality to transform the given equation into an equivalent equation with the same domain and solution set until you can determine the solution set by inspection (as $x=4$ for (2)c). What are the following properties (let a, b, c, be real numbers):
   a) Addition property of equality?
      (If $a = b$, then $a+c = b+c$);
   b) Subtraction property of equality?
      (If $a = b$, then $a-c = b-c$);
   c) Multiplication property of equality?
      (If $a = b$, then $ac = bc$);
   d) Division property of equality?
      (If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$).

e) Substitution property? (If a = b, then in any expression a may be replaced by b or b may be replaced by a.)

4. Solve, check, and justify each step (domain is set of reals):

$\frac{2}{3}x-5 = 17$

$\frac{2}{3}x-5+5 = 17+5$ Addition property of equality.

$\frac{2}{3}x = 22$ Inverse and identity properties of multiplication.

$\frac{3}{2}\cdot\frac{2}{3}x = \frac{3}{2}\cdot 22$ Multiplication property of equality.

$x = 33$ Inverse and identity properties of multiplication.

Check $\frac{2}{3}\cdot 33-5 \stackrel{?}{=} 17$

$22-5 \stackrel{?}{=} 17$

$17 = 17$

The solution set is $\{17\}$.

5. Solve for one of the variables (C) in terms of the other variable (F). Justify each step. Check.

$F = \frac{9}{5}C+32$

$F-32 = \frac{9}{5}C+32-32$ Subtraction property of equality

$F-32 = \frac{9}{5}C$ inverse and identity properties of addition

$\frac{5}{9}(F-32) = \frac{5}{9}\cdot\frac{9}{5}C$ multiplication property of equality

$\frac{5}{9}(F-32) = C$ inverse and identity properties of multiplication.

Check

$F \stackrel{?}{=} \frac{9}{5}\cdot\frac{5}{9}(F-32)+32$

$F \stackrel{?}{=} (F-32)+32$

$F = F$

APPLICATIONS:

1. Solve and check (domain is set of reals):
   - a) $\frac{x-7}{6} = \frac{x-8}{2}$
   - b) $0.3x-.2 = 0.7x+1.4$
   - c) $\frac{x-4}{4} = \frac{x-3}{3} - \frac{x-4}{8}$

2. Solve for the indicated variable:
   - a) $S = \frac{n}{2}(a+\ell)$; n.
   - b) $S = \frac{1}{2}at^2+V_0 \cdot t$; a.

3. List the elements of the solution set:
   - a) $\{x \mid x+7 = 21 \text{ and } x \in \text{counting numbers}\}$;
   - b) $\{x \mid 5-3x = 4x \text{ and } x \in \text{integers}\}$;
   - c) $\{x \mid 2(x+7)-10 = 5x \text{ and } x \in \text{rational numbers}\}$.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) In a proportion, the product of the means equals the product of the extremes.

   $2(x-7) = 6(x-8)$. $2x-14 = 6x-48$. $2x-14+48 = 6x-48+48$.

   $2x+34 = 6x$. $-2x+2x+34 = -2x+6x$. $34 = 4x$

   $\frac{1}{4}.34 = \frac{1}{4}.4x$; $\frac{34}{4} = x$; $\frac{17}{2} = x$; Check $\frac{\frac{17}{2}-7}{6} \stackrel{?}{=} \frac{\frac{17}{2}-8}{2}$

   $\frac{\frac{3}{2}}{6} \stackrel{?}{=} \frac{\frac{1}{2}}{2}$; $\frac{3}{2}.2 \stackrel{?}{=} \frac{1}{2}.6$; $3=3$.

   b) $0.3x-.2-1.4 = 0.7x+1.4-1.4$; $0.3x-1.6 = 0.7x$;

   $-0.3x+0.3x-1.6 = -0.3x+0.7x$; $-1.6 = 0.4x$.

   $\frac{-1.6}{0.4} = \frac{0.4x}{0.4}$; $-4 = x$; Check $0.3(-4)-.2 \stackrel{?}{=} 0.7(-4)+1.4$.

   $-1.2-.2 \stackrel{?}{=} -2.8+1.4$; $-1.4 = -1.4$.

   c) Multiply both sides of the equation by the least common denominator of 4, 3, and 8, which is 24.

   $24(\frac{x-4}{4}) = 24(\frac{x-3}{3} - \frac{x-4}{8})$; $6(x-4) = 8(x-3)-3(x-4)$.

   $6x-24 = 8x-24-3x+12$; $6x-24 = 5x-12$; $x=12$.

   Check $\frac{12-4}{4} \stackrel{?}{=} \frac{12-3}{3} - \frac{12-4}{8}$; $\frac{8}{4} \stackrel{?}{=} \frac{9}{3} - \frac{8}{8}$; $2 \stackrel{?}{=} 3-1$; $2=2$.

2. a) $2S = n(a+\ell)$. $\frac{2S}{a+\ell} = n$
   b) $S-V_0 \cdot t = \frac{1}{2}at^2$. $2(S-V_0 \cdot t) = at^2$.
   c) $\frac{2(S-V_0 \cdot t)}{t^2} = a$

3. a) $x=14$. The solution set is $\{14\}$.
   b) The solution set is $\emptyset$.
   c) $2x+14-10 = 5x$; $4 = 3x$. $\frac{4}{3}$; $= x$.
   Solution set is $\left\{\frac{4}{3}\right\}$.

SUMMARY:

1. What are open sentences, closed sentences, identities, and conditional equations?
2. What are the postulates of equality?
3. What is the substitution postulate?

LESSON: 4

AIM: How do we solve verbal problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Choose a variable and state what the variable represents.
2. Write an equation involving that variable.
3. Solve the equation.
4. Check using the words of the problem.

VOCABULARY: variable, consecutive integers

MOTIVATION:

If x represents a real number, what are the algebraic translations of the following expressions:

a) 15 more than x;
b) x increased by 9;
c) 4 less than x;
d) x decreased by 7;
e) twice as much as x;
f) the product of 11 and x;
g) the average of 10 and x;
h) the quotient of x divided by 3?

DEVELOPMENT:

1. The answers to the motivation problem are:
   a) $x+15$; b) $x+9$; c) $x-4$; d) $x-7$; e) $2x$; f) $11x$; g) $\frac{10+x}{2}$;
   h) $\frac{x}{3}$. In solving verbal problems, it is often necessary to translate words into algebraic expressions, as was done for the motivation problem.

2. Problem solving is one of the most important mathematical skills. What overall problem solving strategy (that is, what specific steps) would you use to solve the following verbal problem:
   a) (number theory). The sum of three integers is 70. The greatest integer is two more than twice the least

integer. The remaining integer is eight more than the least integer. What are the three integers?

b) (distance). Two planes leave JFK airport and travel in opposite directions. After 3 hours the planes are 3,720 Km apart. If the rate of one of the planes is 40 Km more than the rate of the other, what are the rates of the two planes?

c) (mixture). Type A coffee beans are worth 80¢ per pound and type B coffee beans are worth 70¢ per pound. How many pounds of each type are needed to make a 40 pound mixture which is worth 78¢ per pound?

d) (geometry). The perimeter of a rectangle is 72 cm. If the length of the rectangle is two more cm than the width, what is the area of the rectangle (in sq. cm)?

3. Have class recall that a possible problem solving strategy to solve the verbal problems is:
   a) read the problem carefully;
   b) determine what is given and what you must find;
   c) let the variable represent what you must find;
   d) determine what formulas (distance, mixture, geometry) could apply;
   e) possibly set up a chart (distance, mixture) or diagram (geometry);
   f) translate the words into algebraic expressions (number theory);
   g) write an equation;
   h) solve the equation;
   i) check the answer using the original words of the problem;
   j) write the answer using the correct units.

4. Use the problem solving strategy to solve the four verbal problems from (2).

   a) let x = least integer
   $2x+2$ = greatest integer
   $x+8$ = remaining integer

   Since the sum of the three integers is 70, solve the equation.

$$\begin{aligned} x+2x+2+x+8 &= 70 \\ 4x+10 &= 70 \\ 4x &= 60 \\ x &= 15. \quad 2x+2 = 32. \quad x+8 = 23. \end{aligned}$$

Check: The sum of 15, 32, and 23 is 70. The greatest integer (32) is two more than twice the least integer (15). The remaining integer (23) is 8 more than the least integer.

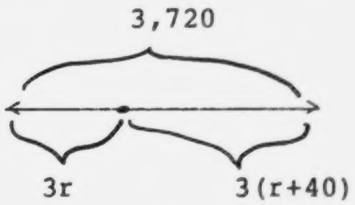
b) Let r = the rate (in Km) of the slower plane.

$r+40$ = the rate (in Km) of the faster plane.

The formula that applies is rate·time = distance.

A chart and a diagram might be helpful.

	rate	· time	= distance
slower plane	r	3	$3r$
faster plane	$r+40$	3	$3(r+40)$
total distance			3,720



Since the planes are going in opposite directions, their distances are added.

$$3r+3(r+40) = 3720$$

Solve:
$$\begin{aligned} 3r+3r+120 &= 3720 \\ 6r+120 &= 3720 \\ 6r &= 3600 \\ r &= 600 \text{ Km - slower plane} \\ r+40 &= 640 \text{ Km - faster plane} \end{aligned}$$

Check: $3\cdot(600)+3(640) = 3720$

c) Let x = number of pounds of 80¢ per pound coffee.

$40-x$ = number of pounds of 70¢ per pound coffee.

The formula that applies is:

(price per pound)·(number of pounds) = total value

A chart might be helpful.

Price per pound	· number of pounds	= total value
80	x	$80x$
70	$40-x$	$70(40-x)$
78	40	$78(40)$

Since the two types of beans form a mixture, the sum of total values of the two types equals the total value of the mixture.

$80x+70(40-x) = 78(40)$

Solve: $80x+2800-70x = 3120$

$10x = 320$

$x = 32$ pounds of 80¢ coffee

$40-x = 8$ pounds of 70¢ coffee

Check: The total value of the 80¢ coffee is $32.80 or $25.60. The total value of the 70¢ coffee is $8.70 or $5.60. $25.60+$5.60 is $31.20, which is 40·78.

d) Let w = width (in cm.) of the rectangle.
w+2 = length (in cm.) of the rectangle.

The formula that applies is:

Perimeter = 2·length+2·width.

$72 = 2(w+2)+2w$. Solve $72 = 2w+4+2w$.

$72 = 4w+4$. $68 = 4w$. $w = 17$cm. $w+2 = 19$cm.

Area = length·width. Area = $19 \cdot 17 = 323$ sq. cm.

Check: 19 is two more than 17. Perimeter equals $2 \cdot 17+2 \cdot 19$, which is 72.

APPLICATIONS:

1. Find three consecutive odd integers such that three times the third integer plus the first integer is 379 more than the second integer.
2. Two trains started at the same time from cities that are 435 miles apart and traveled towards each other. The trains met in five hours. If one train went 5 mph faster than the other train, what were the average speeds?
3. Fran invested one sum of money at 8% and half as much money at 10%. If the first investment pays $300 more in interest per year, how much money was invested at each rate?
4. Find the measure of an angle of a triangle if its measure is 30 less than the sum of the measures of the other two angles.
5. The sum of the ages of Teri and her mother are 47. In 17 years Teri's mother will be twice as old as Teri will be. How old are Teri and her mother now?

SOLUTIONS TO APPLICATION PROBLEMS:

1. Let x, $x+2$, $x+4$ be three consecutive odd integers.
   $3(x+4)+x = x+2+379$. $3x+12+x = x+381$. $3x = 369$. $x = 123$.
   $x+2 = 125$. $x+4 = 127$.

2. Let r = rate of slower (mph), $r+5$ = rate of faster (mph).
   $5r+5(r+5) = 435$. $5r+5r+25 = 435$. $10r = 410$.
   $r = 41$ mph. $r+5 = 46$ mph.

3. Let x = amount of money at 8%, $\frac{x}{2}$ = amount of money at 10%. Interest = principal·rate·time. For one year, time = 1. $.08x = \frac{.10x}{2} + 300$. $.08x = .05x+300$. $.03x = 300$.
   $x = \$10,000$ at 8%. $\frac{x}{2} = \$5,000$ at 10%.

4. Let x = measure of one angle, $x+30$ = sum of measures of other two angles. $x+x+30 = 180$. $2x = 150$. $x = 75$.

5. Let x = Teri's age now, $47-x$ = her mother's age now, $x+17$ = Teri's age in 17 years, $47-x+17$ = her mother's age in 17 years.

   $$47-x+17 = 2(x+17)$$
   $$64-x = 2x+34$$
   $$30 = 3x$$
   $$10 = x \quad 47-x = 37$$

   Teri is 10 years old now. Her mother is 37 years old now.

SUMMARY:

1. What are the steps in a problem solving strategy for verbal problems?

2. What are formulas for distance and mixture problems?

LESSON: 5

AIM: How do we solve inequalities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the properties of inequalities.
2. Solve inequalities (linear) and plot solution sets on the the number line.

VOCABULARY: inequalities, linear inequalities, properties of order, comparison property, transitive property

MOTIVATION:

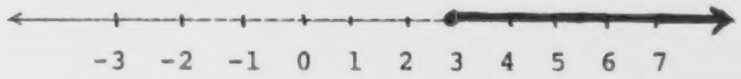
What are the inequalities and graphs for:

a) x is greater than or equal to 3;
b) x is less than or equal to 3;
c) x is greater than 3;
d) x is less than 3;
e) x does not equal 3?

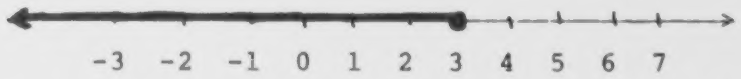
DEVELOPMENT:

The inequalities and graphs for the motivation problem are:

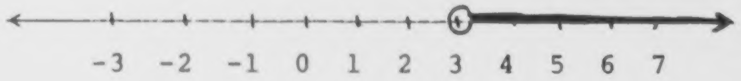
a) $x \geq 3$



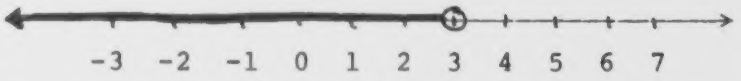
b) $x \leq 3$



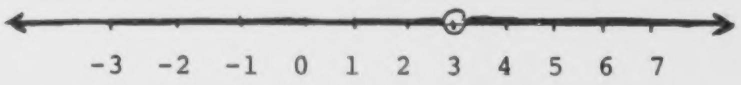
c) $x > 3$



d) $x < 3$



e) $x \neq 3$



What are inequalities? Inequalities are mathematical sentences that contain the symbols $\geq$, $\leq$, $>$, $<$, or $\neq$. Note that in plotting the graphs, a closed circle is used for $\geq$ and $\leq$, while an open circle is used for $>$, $<$, and $\neq$.

2. How do you solve $5(x+3)<25$? Justify each step.

$5x+15<25$	distributive property
$5x+15-15<25-15$	addition property of inequality
$5x<10$	additive inverse, identity
$\frac{1}{5}\cdot 5x<\frac{1}{5}\cdot 10$	multiplication property of inequality
$x<2$	multiplicative inverse, identity

Solution set is $\{x \mid x<2 \text{ and } x\in R\}$.

How would you check the solution set? Since the solution set is infinite, you can not check every value in the set. You can, however, check one real number less than 2 in the original problem. For example, choose $x=1$. Then $5(1+3)<25$, since $20<25$.

3. How do you solve $4(2-x)>12$? Justify each step.

$8-4x>12$	distributive property
$-8+8-4x>-8+12$	addition property of inequality
$-4x>4$	additive inverse, identity
$(-\frac{1}{4})\cdot(-4x)<(-\frac{1}{4})\cdot 4$	multiplication property of inequality
$x<-1$	multiplicative inverse, identity

Note that if you multiply or divide an inequality by a negative number, the sense of the inequality changes from > to <, or from < to >.

4. There are two properties of order which allow us to compare two real numbers. Assume a, b, and c are real numbers.
   a) What is the trichotomy property? One and only one of the following is true: $a<b$, or $a=b$, or $a>b$.
   b) What is the transitive property of inequalities? If $a>b$ and $b>c$, then $a>c$. Also, if $c<b$ and $b<a$, then $c<a$.

5. In order to solve the first degree or linear inequalities that we did, we used postulates of inequalities. Using a, b, and c to represent real numbers, state the following properties of inequalities for >:
   a) Addition. If $a>b$, then $a+c > b+c$;
   b) Subtraction. If $a>b$, then $a-c > b-c$;
   c) Multiplication. If $a>b$ and $c>0$, then $ac > bc$;
   If $a>b$ and $c<0$, then $ac < bc$;

d) Division. If $a>b$ and $c>0$, then $\frac{a}{c} > \frac{b}{c}$;
If $a>b$ and $c<0$, then $\frac{a}{c} < \frac{b}{c}$.

APPLICATIONS:

1. Using the symbol >, arrange the given numbers in the proper order:
   a) $-\frac{3}{8}, -\frac{5}{6}, -\frac{4}{7}$;
   b) .37, .371, .3689

2. List the elements of the solution set:
   a) $\{x \mid x-5<2$ and x counting numbers$\}$;
   b) $\{x \mid 7(3+x) \le 35$ and x whole numbers$\}$.

3. Solve (domain is set of real numbers) and plot the solution set:
   a) $-7 \ge \frac{x}{2} - 9$;
   b) $\frac{x}{3} - \frac{3}{4} > \frac{x}{4}$;
   c) $2(x-12) < 4(3-2x)+5(7x-6)$
   d) $13 - 0.69x \ge x$

* 4. I am thinking of a set of real numbers. If any number is chosen from this set and tripled, the product is greater than or equal to one-half the chosen number minus 10. What is the set of numbers?

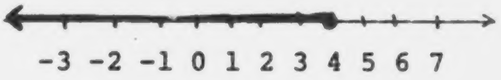
* 5. I am thinking of my age (to the nearest integer). The sum of one-third my age plus one-fifth my age is less than 16. What are possible ages for me?

SOLUTIONS TO APPLICATION PROBLEMS:

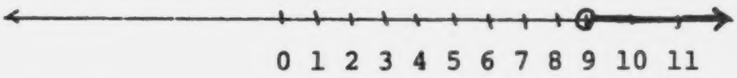
1. a) $-\frac{3}{8} > -\frac{4}{7} > -\frac{5}{6}$. Try converting to decimals.
   b) $.371 > .37 > .3689$

2. a) $x<7$. $\{1,2,3,4,5,6\}$
   b) $21+7x \le 35$. $7x \le 14$. $x \le 2$. $\{0,1,2\}$.

3. a) $2 \ge \frac{x}{2}$. $4 \ge x$. $\{x \mid x \le 4\}$.



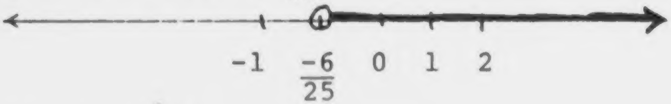
   b) $\frac{x}{3} - \frac{x}{4} > \frac{3}{4}$. $\frac{4x-3x}{12} > \frac{3}{4}$. $\frac{x}{12} > \frac{3}{4}$. $x > 9$.
   $\{x \mid x>9\}$.



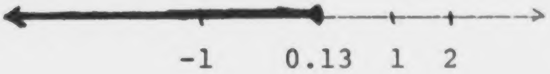
c) $2x-24 < 12-8x+35x-30$

$2x-24 < -18+27x$. $-6 < 25x$. $-\frac{6}{25} < x$.

$\{x \mid x > -\frac{6}{25}\}$.



d) $13 \geq 1.69x$ $.13 \geq x$ $\{x \mid x \leq .13\}$



4. Let x = chosen number. $3x \geq \frac{x}{2} - 10$
$3x - \frac{x}{2} \geq -10$. $\frac{5x}{2} \geq -10$. $x \geq -4$. $\{x \mid x \geq -4\}$.

5. Let x = my age. $\frac{x}{3} + \frac{x}{5} < 16$. $\frac{5x+3x}{15} < 16$.
$8x < 240$. $x < 30$. $\{x \mid x<30 \text{ and } x \in \text{whole numbers}\}$. To put a more reasonable lower bound on my age than 0, we'll need the next lesson (on compound inequalities).

SUMMARY:

1. What is an inequality?
2. What are the properties of order?
3. What are the postulates of inequalities?

LESSON: 6

AIM: How do we solve compound inequalities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use conjunction and disjunction to solve compound inequality sentences.
2. Graph solution sets to compound inequality sentences.

VOCABULARY: compound inequalities, conjunction, conjunctive inequality, disjunction, disjunctive inequality

MOTIVATION:

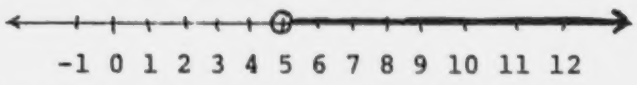
A real number is greater than 5 and less than or equal to 9. Write and graph the solution set.

DEVELOPMENT:

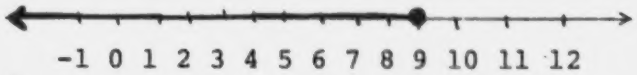
1. The motivation problem is a conjunctive inequality, a mathematical sentence consisting of two inequalities with the word "and". A conjunction is true if and only if both of its clauses are true. The motivation problem may be written in one of the following ways:
   a) $5<x$ and $x\leq 9$;
   b) $5<x \wedge x\leq 9$;
   c) $5<x\leq 9$.

   The intersection of the solution sets of the two inequalities is the solution set of the conjunctive inequality. Therefore, the solution set is $\{x:5<x\} \wedge \{x:x\leq 9\} = \{x:5<x\leq 9\}$. The graph is the solution set of the conjunctive inequality is the intersection of the graphs of the solution sets of the two inequalities.

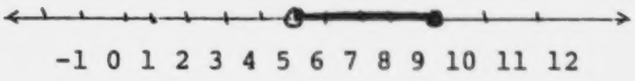
$\{x:5<x\}$



$\{x:x\leq 9\}$



$\{x:5<x\leq 9\}$



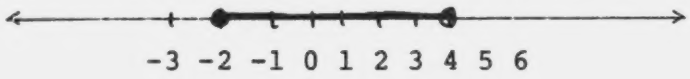
2. Solve and graph the solution set:
$5 \leq 4x-3 < 13$. Solve the inequalities separately and find the intersection of their solution sets.

$5 \leq 4x-3 \wedge 4x-3<13$

$8 \leq 4x \quad \wedge \quad 4x<16$

$2 \leq x \quad \wedge \quad x<4$

The solution set is $\{x: 2 \leq x < 4\}$. The graph of the solution set is:



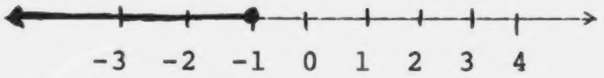
3. A real number is less than or equal to -1 or greater than 3. Write and graph the solution set. This is an example of a disjunctive inequality, a mathematical sentence consisting of two inequalities with the word "or". A disjunction is true if and only if at least one of its clauses is true. Conjunctive inequalities and disjunctive inequalities, which consists of two inequalities, are compound inequalities. The disjunctive inequality may be written in one of the following ways:

a) $x \leq -1$ or $x>3$;

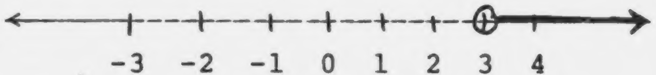
b) $x \leq 1 \wedge x>3$.

The union of the solution sets of the two inequalities is the solution set of the disjunctive inequality. Therefore, the solution set is $\{x: x \leq -1\} \cup \{x: x>3\} = \{x: x \leq -1 \vee x>3\}$. The graph of the solution set of the disjunctive inequality is the union of the graphs of the solution sets of the two inequalities.

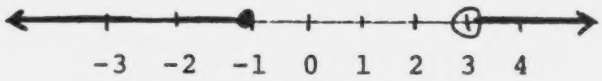
$\{x: x \leq -1\}$



$\{x: x>3\}$



$\{x: x < \underline{x} -1 \quad x>3\}$



4. Solve and graph the solution set:
$3x-2<4 \vee -2x+5 \leq -7$. Solve the inequalities separately and find the union of their solution sets.

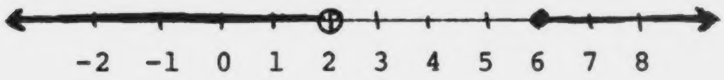
$3x-2<4 \vee -2x+5\le -7$

$3x<6 \quad \vee -2x\le -12$

$x<2 \quad \vee \ x\ge 6$

The solution set is $\{x: x<2 \vee x\le 6\}$.

The graph of the solution set is:



APPLICATIONS:

1. What is the solution set of:
   - a) $x<-3 \wedge x<2$
   - b) $x<4 \vee x<-6$
   - c) $3>x>8$
   - d) $-6<7+x<12$
   - e) $4(x-2)<8x-4<2(x+1)$
   - f) $\frac{3}{5}(3-x)<\frac{7-x}{5}<\frac{4}{5}(6-x)$
   - g) $x<-5 \wedge x>3$

* 2. In $\triangle XYZ$, $XY = 7$ and $YZ = 11$. What statement can be made about the length of $\overline{XZ}$?

* 3. What set(s) of four consecutive even integers satisfy the condition that their average is greater than one-third the second integer and less than one-third the greatest integer?

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\{x: x<-3\}$
   - b) $\{x: x<4\}$
   - c) $\{x: 3>x>8\}$
   - d) $-13<x<5$; $\{x: -13<x<5\}$
   - e) $4x-8<8x-4 \wedge 8x-4<2x+2$

     $-4<4x \quad \wedge \quad 6x<6$

     $-1<x \quad \wedge \quad x<1$

     $\{x: -1<x<1\}$

f) $\frac{3}{5}(3-x)<\frac{7-x}{5} \wedge \frac{7-x}{5}<\frac{4}{5}(6-x)$

$3(3-x)<7-x \wedge 7-x<4(6-x)$

$9-3x<7-x \wedge 7-x<24-4x$

$2<2x \wedge 3x<17$

$1<x \wedge x<\frac{17}{3}$

$\left\{x:1<x<\frac{17}{3}\right\}$

g) $\emptyset$

2. The Triangle Inequality states that the sum of the lengths of two sides of a triangle is greater than the length of the third side.

XY+YZ>XZ and XY+XZ>YZ and XZ+YZ>XY

7+11>XZ and 7+XZ>11 and XZ+11>7

18>XZ and XZ>4 and XZ>-4

4<XZ<18 (if XZ>4, then XZ>-4).

*3. Let x, x+2, x+4, x+6 be the four consecutive even integers.

$\frac{1}{3}(x+2)<\frac{x+(x+2)+(x+4)+(x+6)}{4}<\frac{1}{3}(x+6)$

$\frac{1}{3}(x+2)<\frac{4x+12}{4} \wedge \frac{4x+12}{4}<\frac{x+6}{3}$

$\frac{1}{3}(x+2)<x+3 \wedge x+3<\frac{x+6}{3}$

$x+2<3x+9 \wedge 3x+9<x+6$

$-7<2x \wedge 2x<-3$

$-\frac{7}{2}<x \wedge x<-\frac{3}{2}$

$-\frac{7}{2}<x<-\frac{3}{2}$

The only possible choice for x is -2. The corresponding set of consecutive even integers is $\{-2,0,2,4\}$.

To check, $\frac{1}{3}\cdot 0 \overset{?}{<} \frac{-2+0+2+4}{4} \overset{?}{<} \frac{1}{3}\cdot 4$

$0<1<\frac{4}{3}$

SUMMARY:

1. What are conjunctive inequalities and disjunctive inequalities?
2. How are the solution sets determined for conjunctive inequalities and for disjunctive inequalities?

LESSON: 7

AIM: How do we solve equations involving absolute values?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the definition of absolute values.
2. Solve equations involving absolute values.
3. Plot solution sets on the real number line.

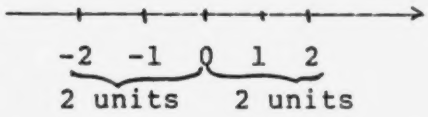
VOCABULARY: absolute value, extraneous root

MOTIVATION:

In manufacturing a one foot long stick, a company states that the length of the stick might vary from one foot by as much as .3 inches. Let x equal the length of the stick. Write an equation to find the largest and smallest possible values of the lengths of the stick.

DEVELOPMENT:

1. To solve problems such as the motivation problem, which involve positive deviations from intended measurements, we need the concept of absolute value. Geometrically, the absolute value of a number, written as $|x|$, represents the distance of the number from 0 on the number line. For example, if $|x|=2$, then we get on the number line:

   Therefore, $x=2$ or $x=-2$. The solution set is $\{2,-2\}$. The absolute value of a real number, x, is defined as:

   $|x| = x$ if $x \geq 0$

   $|x| = -x$ if $x < 0$

   Thus, $|7|=7$ and $|-7|=7$, since the solution set of $|x|=7$ is $\{7,-7\}$.

2. For the motivation problem, $|x-12|=.3$, since the deviation of .3 is from 12. How can we use the definition of absolute value to write two equations?

Elicit:

$x-12=.3$ if $x-12\geq 0$ (that is, $x\geq 12$) or $x-12=-.3$ if $x-12<0$ (that is, if $x<12$).

$x = 12+.3$ $\qquad$ $x = 12-.3$

$x = 12.3$ largest possible value $\qquad$ $x = 11.7$ smallest possible value

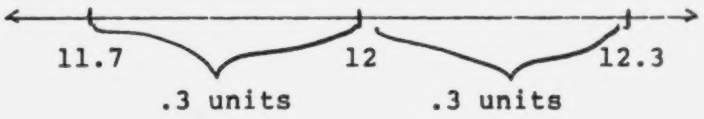
Check each root in the original equation:

$|12.3-12| \overset{?}{=} .3$ $\qquad$ $|11.7-12| \overset{?}{=} .3$

$|.3| \overset{?}{=} .3$ $\qquad$ $|-.3| \overset{?}{=} .3$

$.3 = .3$ $\qquad$ $.3 = .3$

Graphically:



3. What are the steps used in solving equations with absolute values?
   a) Isolate the absolute value expression on one side of the equation.
   b) Use the definition of absolute value to write two equations.
   c) Solve the two equations.
   d) Check the roots of the two equations in the original absolute value equation.

4. For example, solve $2|x|-4 = 3x$.
   a) $2|x| = 3x+4$ $\qquad$ $|x| = \frac{3}{2}x+2$
   b) $x = \frac{3}{2}x+2$ if $x\geq 0$ or $x = -\frac{3}{2}x-2$ if $x<0$
   c) $-\frac{1}{2}x = 2$ or $\frac{5}{2}x = -2$

      $x = -4$ or $x = -\frac{4}{5}$
   d) Check: $x = -4$ $\qquad$ $2|-4|-4 \overset{?}{=} 3(-4)$

      $8-4 \overset{?}{=} -12$

      $-4 \neq -12$ $\quad x = -4$ is an extraneous root (that is, it is not a root of the absolute value equation).

      Check: $x = -\frac{4}{5}$ $\qquad$ $2\left|-\frac{4}{5}\right|-4 \overset{?}{=} 3\left(-\frac{4}{5}\right)$

      $\frac{8}{5} - 4 \overset{?}{=} -\frac{12}{5}$

      $\frac{8}{5} - \frac{20}{5} \overset{?}{=} -\frac{12}{5}$

      $-\frac{12}{5} = -\frac{12}{5}$

      The solution set $= \left\{-\frac{4}{5}\right\}$.

APPLICATIONS:

1. Solve:
   a) $|-x| = 10$
   b) $x^2 = 16$
   c) $\left|\frac{7x-3}{12}\right| = 4$
   d) $|3y-10|+6y = 4$

2. Solve and graph the solution set: $|7w-14| = 21$

3. Under what conditions is $|x| = -x$?

4. Evaluate if $x = -3$:
   a) $4|-x|-|x|$
   b) $|x^2|-x^2$

5. Using absolute value, write an equivalent sentence:
   $x = 4 \vee x = -4$

6. Which of the following equations has the empty set as its solution set?
   a) $|x| = 5$
   b) $|-x| = 5$
   c) $|x| = -5$
   d) $|x-5| = 0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $-x = 10$ or $-x = -10$
   $x = -10$ or $x = 10$
   Solution set is $\{10, -10\}$.
   b) $x = 4$ or $x = -4$ Solution set is $\{4, 4\}$.
   c) $\frac{7x-3}{12} = 4$ or $\frac{7x-3}{12} = -4$

$7x-3 = 48$ $\qquad$ $7x-3 = -48$

$7x = 51$ $\qquad$ $7x = -45$

$x = \frac{51}{7}$ $\qquad$ $x = -\frac{45}{7}$

Check: $\left|\frac{7\cdot\frac{51}{7}-3}{12}\right| \overset{?}{=} 4$ $\qquad$ $\left|\frac{7\cdot\frac{-45}{7}-3}{12}\right| \overset{?}{=} 4$

$\left|\frac{51-3}{12}\right| \overset{?}{=} 4$ $\qquad$ $\left|\frac{-48}{12}\right| \overset{?}{=} 4$

$\frac{48}{12} = 4$ $\qquad$ $\frac{48}{12} = 4$

Solution set is $\left\{\frac{51}{7}, -\frac{45}{7}\right\}$

d) $|3y-10| = 4-6y$

$$3y-10 = 4-6y \quad \text{or} \quad 3y-10 = -4+6y$$

$$9y = 14 \qquad -3y = 6$$

$$y = \frac{14}{9} \qquad y = -2$$

Check: $\left|3\cdot\frac{14}{9}-10\right|+6\cdot\frac{14}{9} \stackrel{?}{=} 4 \qquad |3\cdot(-2)-10|+6(-2) \stackrel{?}{=} 4$

$$\left|\frac{14}{3} - \frac{30}{3}\right|+\frac{28}{3} \stackrel{?}{=} 4 \qquad |-16|-12 \stackrel{?}{=} 4$$

$$\left|-\frac{16}{3}\right|+\frac{28}{3} \stackrel{?}{=} 4 \qquad 16-12 \stackrel{?}{=} 4$$

$$\frac{16}{3}+\frac{28}{3} \neq 4$$ Solution set is $\{-2\}$.

2. $7w-14 = 21$ or $7w-14 = -21$

$$7w = 35 \qquad 7w = -7$$

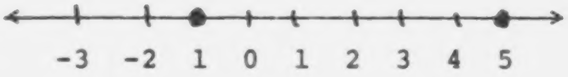
$$w = 5 \qquad w = -1$$

Check: $|7\cdot5-14| \stackrel{?}{=} 21 \qquad |7(-1)-14| \stackrel{?}{=} 21$

$$35-14 = 21 \qquad 21 \stackrel{?}{=} 21$$

$$21 = 21$$

Solution set is $\{5, 1\}$



3. By the definition of absolute value, $|x| = -x$ if $x<0$.

4. a) $4|-(-3)| - |-3| = 4\cdot3-3 = 9$
   b) $|(-3)^2| - (-3)^2 = 9-9 = 0$

5. $|x| = 4$

6. (c). Absolute value can not be negative.

SUMMARY:

1. What is a geometric interpretation of absolute value?
2. What is the definition of absolute value?
3. What steps are needed to solve equations with absolute values?

LESSON: 8

AIM: How are conditional inequalities involving absolute values solved?

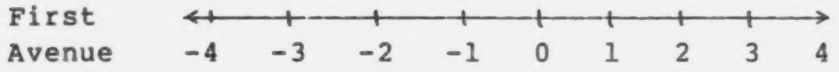
PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve inequalities of the form $|x|<k$ and and $|x|>k$.
2. Plot the solutions on a number line.

VOCABULARY: conditional inequality, conjunctive inequality, disjunctive inequality, absolute value

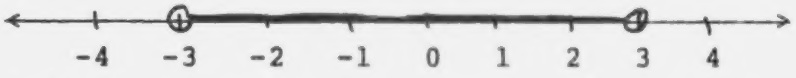
MOTIVATION:

Assume that Juanita is on first avenue and is less than 3 blocks from street 0. Let x = Juanita's position on the number line below. Graph her position and write the corresponding inequality.



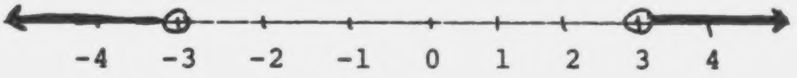
DEVELOPMENT:

1. For the motivation problem, the distance from Juanita to street 0 is less than 3. The absolute value of x is the measure of its distance from 0 on a number line. Therefore, $|x|<3$. The graph is:



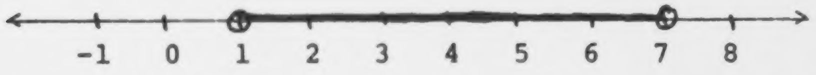
$|x|<3$ is a conjunctive inequality. What is the solution set for $x<3$? The solution set is $\{x: -3<x<3\}$. What is the solution set for $|x|\leq 3$? The solution set is $\{x: -3\leq x\leq 3\}$.

2. Assume Juanita is still on first avenue and her distance from street 0 is greater than 3 blocks. Draw the graph and write the corresponding inequality.



$|x|>3$ is a disjunctive inequality. What is the solution set? The solution set is $\{x: x<-3 \text{ or } x>3\}$. The solution set for $|x| \geq 3$ is $\{x: x \leq -3 \text{ or } x \leq 3\}$. For $|x| \geq k$ (k a real number), the solution set is $\{x: x \leq k \text{ or } x \geq k\}$.

3. If Juanita is on first avenue and is less than 3 blocks from 4th street, what is the graph and corresponding inequality?



Since the distance from 4th street is less than 3 blocks, $|x-4|<3$. To solve the inequality, rewrite the inequality as $-3<x-4<3$ and solve the two derived inequalities:

$-3 < x-4$ and $x-4 < 3$

$+4 \quad +4 \qquad +4 \quad +4$

$1 < x \qquad x < 7$

The solution set is $\{x: 1<x<7\}$.

Solve: $|5+7(x-3)| - 12<0$

$|5+7(x-3)| <12$ isolate the absolute value

$|7x-16|<12$ simplify the absolute value expression

$-12<7x-16<12$

$-12<7x-16$ and $7x-16<12$

$4<7x$ and $7x<28$

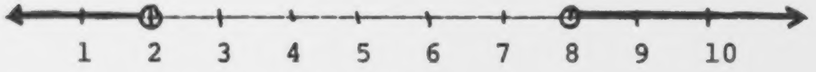
$\frac{4}{7}<x$ and $x<4$ Solution set: $\{x \ \frac{4}{7}<x<4\}$.

4. Solve and graph a disjunctive inequality:

$|x-5|>3$ Rewrite as:

$x-5<-3$ or $x-5>3$

$x<2$ or $x>8$



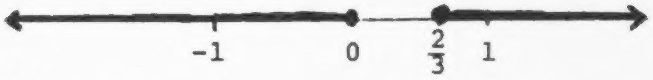
Solve: $|10-6(2x+1)| -4 \geq 0$

$|10-6(2x+1)| \geq 4$ Reverse the sense of the inequality when dividing by a negative number.

$|-12x+4| \geq 4$

The solution set is $\{x: x \leq 0 \text{ or } x \geq \frac{2}{3}\}$.

The graph is:



To check, choose 3 values:

$x=-1\ (x<0)$	$x=\frac{1}{2}\ (0<x<\frac{2}{3})$	$x=1\ (x>\frac{2}{3})$
$\lvert 10-6(2(-1)+1)\rvert -4\overset{?}{\geq}0$	$\lvert 10-6(2.\frac{1}{2}+1)\rvert -4\overset{?}{\geq}0$	$\lvert 10-6(2\cdot 1+1)\rvert -4\overset{?}{\geq}0$
$\lvert 16\rvert -4\overset{?}{\geq}0$	$\lvert 10-12\rvert -4\overset{?}{\geq}0$	$\lvert 10-18\rvert -4\overset{?}{\geq}0$
$12\geq 0$	$-2\not\geq 0$	$4\geq 0$

APPLICATIONS:

1. Write and solve a conjunctive inequality for the following problem: Juanita is on the first avenue and is less than 7 blocks from 5th street. Assume that the value of a position on a street must be positive.

2. Solve: (a) $\lvert \frac{x}{3}+2\rvert <6$

   (b) $\lvert \frac{2x}{5} +10\rvert -20\geq 0$

   (c) $-\frac{3}{4}\lvert 6-3x\rvert +6\leq 0$

* 3. Prove the triangle inequality: for real numbers x and y, $\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$.

* 4. Using the triangle inequality, prove for real numbers a and b: $\lvert a\rvert - \lvert b\rvert \leq \lvert a-b\rvert$.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $x-5 <7 \wedge x>0$

   $-7<x-5$ and $x-5<7$ and $x>0$

   $-2<x$ and $x<12$ and $x>0$

   $\{x\mid 0<x<12\}$.

2. a) $-6<\frac{x}{3}+2$ and $\frac{x}{3}+2<6$

   $-8<\frac{x}{3}$ and $\frac{x}{3}<4$

   $-24<x$ and $x<12$

   $\{x\mid -24<x<12\}$.

b) $\left|\frac{2x}{5}+10\right| \geq 20$

$\frac{2x}{5}+10 \leq -20$ or $\frac{2x}{5}+10 \geq 20$

$\frac{2x}{5} \leq -30$ or $\frac{2x}{10} \geq 10$

$x \leq -75$ or $x \geq 25$

$\{x \mid x \leq -75 \text{ or } x \leq 25\}$.

c) $-\frac{3}{4}|6-3x| \leq -6$

$|6-3x| \geq 8$

$6-3x \leq -8$ or $6-3x \geq 8$

$-3x \leq -14$ or $-3x \geq 2$

$x \geq \frac{14}{3}$ or $x \leq -\frac{2}{3}$

$\{x \mid x \leq -\frac{2}{3} \text{ or } x \leq \frac{14}{3}\}$.

3. $-|x| \leq x \leq |x|$ since $|x|$ is nonnegative and $-|x|$ is nonpositive.

$-|y| \leq y \leq |y|$ add the two conjunctive inequalities to get:

$-|x|-|y| \leq x+y \leq |x|+|y|$ rewrite the conjunctive inequality as:

$|x+y| \leq |x|+|y|$

4. Let $a=x+y$, $b=y$. From the triangle inequality,

$|a| \leq |a-b|+|b|$

$|a|-|b| \leq |a-b|$

SUMMARY:

1. What is the solution set for $|x| \leq k$ (k is a real number)?
2. What is the solution set for $|x| \geq k$?
3. What are the steps needed to solve inequalities of the form $|ax+b|+c \geq 0$ (where $a, b, c \in R$)?

LESSON: 9

AIM: What are the laws of exponents?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the five laws of exponents.
2. Simplify expressions using the laws of exponents.
3. Simplify expressions containing zero and negative exponents.

VOCABULARY: exponents, coefficient, power, positive integer, base.

MOTIVATION:

1. What is the value of $(3K^3) \cdot (4K)^0$?

DEVELOPMENT:

1. Elicit that in order to do this we must know the Laws of exponents and the value of $(4K)^0$. Elicit that in the expression $3K^3$ each element has a name. The 3 in front is called the coefficient, the K is the base and the raised 3 is the exponent. (The students should know this) Let us consider a simpler example - $(3K^3)(2K^5)$ Elicit that this is $6K^8$ since we multiply the coefficients, keep the base and add the exponents when multiplying such expressions. The general rule $x^a \cdot x^b = x^{a+b}$ should also be elicited and written on the board.
2. Consider $x^3 \cdot x^3 \cdot x^3 \cdot x^3$. What would be an equivalent expression in simplified form? From above this equals $x^{3+3+3+3} = x^{12}$. How could we write $x^3 \cdot x^3 \cdot x^3 \cdot x^3$ in a different way? Elicit $(x^3)^4$. Then $(x^3)^4 = x^{12}$. Hence the second rule of exponents is $(x^a)^b = x^{ab}$. That is when raising on exponent to a power we keep the base and multiply the exponents. Ask the class to simplify (a) $(2x^3)^2$ (b) $(4Y^2)^3$ answers: (a) $4x^6$ (b) $64Y^6$

3. This leads us to simplify $(x^6 \cdot Y^3)^2$. This would be $(x^6 \cdot Y^3)(x^6 \cdot Y^3) = x^{12} \cdot Y^6$. This could have been derived from multiplying the exponents in the parenthesis by the power outside. Hence we have $(x^a \cdot Y^b)^r = x^{ar} \cdot Y^{br}$.

4. What would $\frac{y^5}{y^2}$ equal? By expanding this we get $\frac{\overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot y \cdot y \cdot y}{\underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}}}$ and dividing we get y^3. Therefore the rule is, that when we divide expressions with the same base we keep the base and subtract the exponents. That is $\frac{x^a}{x^b} = x^{a-b}$.

Simplify $\frac{6x^5}{2x^3}$: What do we do with the coefficient? (Divide them.) Hence $\frac{6x^5}{2x^3} = 3x^2$

What would $\frac{a^{12}}{a^{12}} = ?$ By the above rule $\frac{a^{12}}{a^{12}} = a^0$. What else does it =? (1) Why?
(Because any number, $\neq 0$, divided by itself = 1). Hence we now know that $a^0 = 1$. Now what does $(4K)^0$ =? $(4k)^0 = 1$ since anything to the zero power (except 0) = 1. Therefore what is $(3k^3)(4k)^0$ =? (answer $3k^3$)

5. How would you simplify $\frac{x^6}{x^8}$? By the law of division this equals $x^{6-8} = x^{-2}$. If we were to expand each we would get $\frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot x \cdot x}$ and by dividing we get $\frac{1}{x^2}$.

This leads us to see that $x^{-2} = \frac{1}{x^2}$.

Therefore we can determine the definition of a negative exponent. That is x^{-n}, where n is a positive integer, is equal to $\frac{1}{x^n}$. From this we look at an example such as $\left(\frac{2a^2}{3b^4}\right)^{-2}$. How can we simplify this expression?

$$\left(\frac{2a^2}{3b^4}\right)^{-2} = \frac{1}{\left(\frac{2a^2}{3b^4}\right)^2} = 1\cdot\left(\frac{3b^4}{2a^2}\right)^2 = \frac{9b^8}{4a^4}$$

Which gives us our final rule of exponents when n is a positive integer.

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n = \frac{y^n}{x^n}$$

APPLICATIONS: Simplify

1. $(3x^2y)(4xy^3)$

2. $(-5a^3b^2)(2a^2b)$

3. $\left(\frac{3x^2y}{2}\right)^3$

4. $\frac{(6ab)(2a^2)}{(-3a^2b^3)^2}$

5. $\left(\frac{4}{3y^2}\right)^{-2}$

6. $\frac{4a^4b^2}{6a^3b^5}$

7. $\left(\frac{3x^5}{5y^2x}\right)^{-3}$

8. $\frac{a^5b^{-2}}{3b^2}$

9. $\frac{5x^0y}{(-x^{-2}y)^3}$

10. $\left(\frac{42x^{-3}b^2}{28x^{-2}c^{-4}}\right)^{-1}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $(3x^2y)(4xy^3) = 12x^3y^4$

2. $(-5a^3b^2)(2a^2b) = -10a^5b^3$

3. $\left(\frac{3x^2y}{2}\right)^3 = \frac{27x^6y^3}{8}$

4. $\frac{(6ab)(2a^2)}{(-3a^2b^3)^2} = \frac{12a^3b}{9a^4b^6} = \frac{3a^3b(4)}{3a^3b(3ab^5)} = \frac{4}{3ab^5}$

5. $\left(\frac{4}{3y^2}\right)^{-2} = \left(\frac{3y^2}{4}\right)^2 = \frac{9y^4}{16}$

6. $\frac{4a^4b^2}{6a^3b^5} = \frac{2a}{3b^3}$

7. $\left(\frac{3x^5}{5y^2x}\right)^{-3} = \frac{5y^2x}{3x^5}\ ^3 = \frac{125y^6x^3}{27x^{15}} = \frac{125y^6}{27x^{12}}$

8. $\frac{a^5\ b^{-2}}{3b^2} = \frac{a^5}{3b^2 \cdot b^2} = \frac{a^5}{3b^4}$

9. $\frac{5x^0y}{(-x^{-2}y)^3} = \frac{5y}{-x^{-6}y^3} = \frac{-5x^6}{y^2}$

10. $\left(\frac{42x^{-3}b^2}{28x^{-2}C^{-4}}\right)^{-1} = \frac{28x^{-2}C^{-4}}{42x^{-3}b^2} = \frac{4x^3}{7x^2b^2C^4} = \frac{4x}{7b^2C^4}$

SUMMARY:

1. When raising an expression such as $(3x^2)$ to a power of 2. (That is $(3x^2)^2$, what happens to the coefficient? To the exponent? (answer $9x^4$).
2. What is an expression with positive exponents that is = to $(2y^2)^{-3}$?

   answer: $(2y^2)^{-3} = \frac{1}{(2y^2)^3} = \frac{1}{8y^6}$

LESSON: 10

AIM: How can we simplify polynomial expressions using addition, subtraction and multiplication?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define monomial, polynomial, coefficient and like terms.
2. Simplify expressions by adding and subtracting polynomials.
3. Multiply binomials and polynomials.

VOCABULARY: monomial, polynomial, binomial

MOTIVATION:

If a sporting goods chain had 3 stores and they wanted to take inventory of the different balls they have in stock, how would they do it?

DEVELOPMENT:

1. Elicit that they would count the number of each type in each store and then add them. Mathematically - what is this equivalent to? (Combining like terms.) What do we call terms such as x, $2x^2$, $3y^2z^3$? (Monomials.) What are terms such as $(2x+3)$, $(3x^2+4x-2)$, $(x^2+2x^2y+4xy^2+3y^2)$? (Polynomials) How do we add polynomials? Elicit we combine like terms. Add $(3x^2+4x-2)$ and $(2x^2-3x+5)$ we get
$$\begin{array}{r}\underline{2x^2-3x+5}\\5x^2+x+3.\end{array}$$
Have the students try $(3x^3y^2+4x^2y^3-2x^2y^2)+(4x^2y^2-6x^3y^2-4x^2y^3) = (-3x^3y^2+2x^2y^2)$.

2. Ask how do we subtract $4-(-3)$? Elicit that it becomes $4+(+3) = +7$. What is $(3x^2)-(5x^2) = 3x^2+(-5x^2) = -2x^2$. Elicit the rule for subtraction? (Change the sign of subtrahend and add.) Let us subtract polynomials.

Try: $(5m^2n+3mn^2-4mn)-(7m^2n-4mn^2+6mn)$. Rewrite as $(5m^2n+3mn^2-4mn)+(-7m^2n+4mn^2-6mn) = -2m^2n+7mn^2-10mn$.
Try: $(5x^2+7xy-4y^2)+(-3x^2+5y^2)-(4xy-3y^2)$. Rewrite as $(5x^2+7xy-4y^2)+(-3x^2+5y^2)+(-4xy+3y^2)$ then add. $2x^2+3xy+4y^2$ is the solution.

3. Multiply $4xy(2x^2y)$? We get $8x^3y^2$. Review multiplication of monomials - (1) multiply coefficients (2) keep bases (3) add exponents. Multiply $3x(5x^2+2x^3-3x^4)$? (Ans. $15x^3+6x^4-9x^5$) Multiply $(x+3)(x-4)$. Review "Foil" or "Double Distributive". We get $x^2+3x-4x-12 = x^2-x-12$. What would we do to multiply $(2n-4)(4n^2-2n+1)$. Elicit multiply 2n times polynomial, then -4 times polynomial to get $8n^3-4n^2+2n-16n^2+8n-4 = 8n^3-20n^2+10n-4$. Have students try $(2x-5y)(x^3-3x^2y+2y^3)$. We get $2x^4-6x^3y+4xy^3-5x^3y+15x^2y^2-10y^4$. Combining terms to get $2x^4-11x^3y+4xy^3+15x^2y^2-10y^4$.

APPLICATIONS:

A. Simplify

1. $(10a+5ab-6b)+(8a-6ab)$
2. $(8x+5y-2z)-(3x-9y+7z)$
3. $(x^2-17+5x)-(x^3-7x)+(4x-5x^3+9)$
4. $3a(a-2b)-4b(2a+b)$

B. Multiply and Simplify

1. $(x+2y)(x-3y)$
2. $(5a+4b)(3a-2b)$
3. $(6x-4y)^2$
4. $(3x-3)(4x^2+3x-1)$
5. $(a-b)(2a^2-3ab+4b^2)$

* 6. (Optional) $(a^2-3ab-b^2)(2a^2-3ab-2b^2)$

SOLUTIONS TO APPLICATION PROBLEMS:

A. 1. $18a-ab-6b$

2. $5x+14y-9z$

3. $-6x^3+x^2+16x-8$

4. $3a^2-14ab-4b^2$

B. 1. $x^2-xy-6y^2$

2. $15a^2+2ab-8b^2$

3. $36x^2-48xy+16y^2$

4. $12x^3-3x^2-12x+3$

5. $2a^3-5a^2b+7ab^2-4b^3$

6. $2a^4-9a^3b+5a^2b^2+9ab^3+2b^4$

SUMMARY:

1. How do we subtract polynomials?
2. How do we multiply a polynomial by a binomial?

LESSON: 11

AIM: How do we factor polynomials?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Factor trinomials of the form ax^2+bx+c completely.
2. Factor trinomial squares of the form $ax^2+2ab+b^2$ and $a^2-2ab+b^2$
3. Factor the difference of two squares of the form a^2-b^2.

* 4. Factor the sum and difference of two cubes of the form a^3+b^3 or a^3-b^3 (optional).

VOCABULARY: factor, trinomial, perfect squares, cubes

MOTIVATION:

I multiplied 2 binomials and got x^2-3x+2, what were the binomials?

DEVELOPMENT:

1. Elicit that to find the binomials we factor the expression x^2-3x+2. We get $(x-2)(x-1)$. Review rules for factoring trinomials into binomial.

   ax^2+bx+c will yield (_x+_)(_x+_)
   ax^2-bx+c will yield (_x-_)(_x-_)
   $ax^2\pm bx-c$ will yield (_x+_)(_x-_)

   and how the numerical values are achieved. Have the class try x^2+3x-4, $y^2-4y-12$ and $8y^2-14x+3$. Discuss that this last example is a little more difficult but the concept and procedure is the same. That is since it is of the form ax^2-bx+c we will begin with the form (_y-_)(_y-_). The coefficients of y must multiply to give 8, the product of the second numbers must equal 3 and the sum of the cross products must equal -14. Since 3 is prime we will have (_y -3)(_y -1). Ask what are the possible coefficients of y. Elicit $8\cdot 1$, $4\cdot 2$. Try these and we see that $(2y-3)(4y-1)$ will multiply to $8y^2-14y-3$

since $(2y)(-1) = -2y$ and $(-3)(4y) = -12y$ and the sum = $-14y$.

Have the class try to factor $12m^4-22m^3+20m^2$. Elicit that $2m^2$ is a common factor so we get $2m^2(6m^2-11m-10)$. Have the students factor the trinomial to achieve $m^2(3m+2)(2m-5)$.

2. Present polynomials x^2-16 and $y^2+10+25$ to be factored. Discuss that these are special in that the first is a difference of two squares and the second a perfect square trinomial. Elicit that the factors of x^2-16 are $(x+4)(x-4)$ and the factors of $y^2+10y+25$ are $(x+5)(x+5) = (x+5)^2$.

   Have the students try more difficult problems such as a) $9x^2-144$ b) $36x^2-81y^2$ and c) $16n^4-1$.

   Elicit the solutions a) $(3x-12)(3x+12)$ b) $(6x+9y)(6x-9y)$ c) $(4n^2-1)(4n^2+1)$. Note: $(4n^2-1)$ can now be similarly factored into $(2n-1)(2n+1)$ so our complete factorization of $16n^4-1 = (2n-1)(2n+1)(4n^2+1)$.

* 3. (Optional) Have the class multiply $(a-b)(a^2+ab+b^2)$. Elicit that the result is a^3-b^3. Therefore how would we factor a^3-b^3? Into $(a-b)(a^2+ab+b^2)$. What would the factors of x^3-y^3 be? $((x-y)(x^2+xy+y^2))$

   Try x^3-8, and y^3-27. (ans. $(x-2)(x^2+2x+4)$ and $(y-3)(y^2+3y+9)$)

   Ask can anyone guess the factors of a^3+b^3? Have the students try their guesses. Then show that $(a+b)(a^2-ab+b^2) = a^3+b^3$. What would be the factors of x^3+1? $((x+1)(x^2-x+1))$.

APPLICATIONS:

Factor each polynomial completely.

1. y^2+6y-7
2. $6x^2-30x+24$
3. $4x^2+4x-3$

4. $16x^2-a^2$

5. $25y^2-49n^2$

6. n^4+2n^2-3

* 7. y^3-27

* 8. $27m^3+64$

9. $4y^3+24y^2-64y$

10. y^4-8y^2+16

SOLUTIONS TO APPLICATION PROBLEMS:

1. $(y+7)(y-1)$

2. $6(x^2-5x+4) = 6(x-4)(x-1)$

3. $(2x+3)(2x-1)$

4. $(4x-a)(4x+a)$

5. $(5y+7n)(5y-7n)$

6. $(n^2+3)(n^2-1) = (n^2+3)(n+1)(n-1)$

* 7. $(y-3)(y^2+3y+9)$

* 8. $(3m+4)(9m^2-12m+16)$

9. $4y(y^2+6y-16) = 4y(y+8)(y-2)$

10. $(y^2-4)^2 = (y+2)^2(y-2)^2$

SUMMARY:

How do we factor 1) $x^2-14x-51$?
* 2) a^3-125 ?

LESSON: 12

AIM: How do we solve quadratic equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve quadratic equations of the form $ax^2+bx+c = 0$ by factoring.
2. Solve quadratic equations using the quadratic formula.
3. Solve verbal problems involving quadratic equations.

* 4. Solve factorable cubic equations of the form $ax^3+bx^2+cx=0$

VOCABULARY: factor, quadratic equation, quadratic formula, coefficient, standard form

MOTIVATION:

Find the width of a rectangle whose length is 2 more than its width and whose area is 48 square cm.

Begin the lesson with a Do Now to review necessary skills. Then present the motivating problem.

Do Now

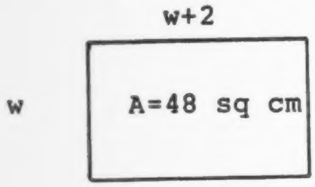
1. factor a) x^2+3x-4 b) $51+u^2-20u$
2. If $a\cdot b=0$, what must be true about a and/or b?

DEVELOPMENT:

1. Review the Do Now.
   Elicit the factors for (1a) x^2+3x-4 are $(x+4)(x-3)$.
   Ask - What would you do to the polynomial $51+u^2-20u$ before you factored it? Elicit - place in descending order of powers. That is $u^2-20u+51$ and the get the factors $(u-17)(u-3)$.
2. Discuss the second Do Now question.
   Have the students determine that a and b are factors and the real question is - if two factors multiply to yield a product of 0, what must be true?
   Elicit that either a, or b, or both must = 0.

Discuss - If a is 0, what could b be? (anything since $0 \cdot n = n \cdot 0 = 0$)

3. With these concepts in place present the motivating problem. Elicit the diagram, the formula and equation needed.



$A = lxw$, $w(w+2) = 48$

Multiply and achieve $w^2+2w = 48$. Discuss - that this is a quadratic equation. Elicit that we usually work with quadratics in another form, that is $w^2+2w-48 = 0$. Elicit that this is called the Standard Form of a quadratic equation and the definition: An equation that can be written in the form $ax^2+bx+c = 0$ where a, b, c are real numbers and $a \neq 0$ is a quadratic equation. Ask for a method of solution. The students should recall that we solve these equations by factoring and therefore the procedure is

$$\begin{array}{c} w^2+2w-48 = 0 \\ (w+8)(w-6) = 0 \end{array}$$

from the Do Now we know that at least one of these factors must be zero therefore we set

$$\begin{array}{lcc} & w+8 = 0 \text{ and } & w-6 = 0 \\ \text{and} & \underline{\quad -8 = -8} & \underline{\quad +6 = +6} \\ \text{solve} & w = -8 & w = +6 \end{array}$$

Ask - can we accept both answers? (No - since we can't have a negative dimension, the width must be = 6.

Have the students solve $x^2-16x+39 = 0$ and $x^2+2x-1 = 0$. Elicit the solution (at the board preferably) to the first as

$$\begin{array}{l} x^2-16x+39 = 0 \\ (x-13)(x-3) = 0 \\ x-13 = 0 \text{ or } x-3 = 0 \\ x = 13 \qquad x = 3 \end{array}$$

Ask - are both acceptable? Elicit that a quadratic equation will have two solutions and show that both check.

4. Elicit that $x^2+2x-1 = 0$ is not factorable and that we have a formula to use. (This should have been derived in Course II.) Elicit the formula as $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and that a, b, and c are the coefficients of the squared term, linear term and constant term respectively in the quadratic equation. In this equation $a = 1$, $b = 2$, and $c = -1$. Substituting we get

$$x = \frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2} = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm2\sqrt{2}}{2} = -1\pm\sqrt{2}.$$

Again have students note that there are two solutions. Have the students solve $2y^2-5y-2 = 0$
Have them determine that $a = 2$, $b = -5$ and $c = -2$ and the solution will be

$$y = \frac{+5\pm\sqrt{(-5)^2-4(2)(-2)}}{2(2)} = \frac{+5\pm\sqrt{25+16}}{4} = \frac{+5\pm\sqrt{41}}{4}$$

5. Present the problem - the sum of twice an integer and the square of the integer is 15. Find the integer. Elicit a legend. Let x = the integer, and the equation $x^2+2x = 15$ and the solution

$$\begin{aligned} x^2+2x-15 &= 0 \\ (x+5)(x-3) &= 0 \\ x+5 = 0 \text{ OR } x-3 &= 0 \\ x = -5 \quad x &= 3 \end{aligned}$$

Ask - can both of these be answers?
Yes. Since $(-5)^2+2(-5) = 25-10 = 15$ and
$(3)^2+2(3) = 9+6 = 15$

6. Present $4x^3+24x^2-64x = 0$. Ask - Is this a quadratic equation? (No) Can we get this to be at least partly a quadratic.

Note: Each term of the equation has 4x as a factor, therefore $4x^3+24x^2-64x = 0$ can be factored to $4x(x^2+6x-16) = 0$. The parenthesis contains a quadratic expression. Therefore we get $4x(x+8)(x-2) = 0$. Ask the class, what must be true if $a\cdot b\cdot c = 0$? Since we have 3 factors we get three solutions. That is $4x = 0$, $x+8 = 0$ and $x-2 = 0$, therefore $x = 0$, $x = -8$, $x = 2$.

APPLICATIONS:

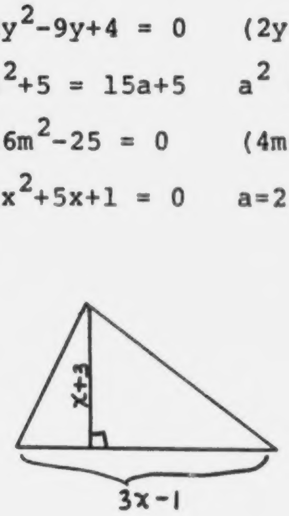
Solve the following equations.

1. $x^2-7x+12 = 0$
2. $2y^2-9y+4 = 0$
3. $a^2+5 = 15a+5$
4. $16m^2-25 = 0$
5. $2x^2+5x+1 = 0$
6. The area of a triangle is 24. If a side of the triangle is 3x-1 and the altitude to that side is x+3, find the length of the altitude.
7. $y^2-6x-3 = 0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $x^2-7x+12 = 0$ $(x-4)(x+3) = 0$ $\{4, 3\}$
2. $2y^2-9y+4 = 0$ $(2y-1)(y-4) = 0$ $\{1/2, 4\}$
3. $a^2+5 = 15a+5$ $a^2 = 15a$ $\{0, 15\}$
4. $16m^2-25 = 0$ $(4m-5)(4m+5) = 0$ $\left\{\pm\frac{4}{5}\right\}$
5. $2x^2+5x+1 = 0$ $a=2, b=5, c=1$ $x = \frac{-5\pm\sqrt{25-8}}{4}$ or $x = \frac{-5\pm\sqrt{17}}{4}$

6.



$\frac{1}{2}((x+3)(3x-1)) = 24$

$3x^2+8x-3 = 48$

$3x^2+8x-51 = 0$

$(3x+17)(x-3) = 0$

$3x = -17$, $x = 3$; altitude = 6

reject

7. $y^2-6x-3 = 0$ $a=1, b=-6, c=-3$ $y = \frac{+6\pm\sqrt{36+12}}{2}$

$$y = \frac{+6\pm\sqrt{48}}{2} = \frac{+6\pm4\sqrt{3}}{2} = 3\pm2\sqrt{3}$$

SUMMARY:

How do we solve a quadratic equation?

LESSON: 13

AIM: How do we divide polynomials?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Divide a polynomial by a monomial.
2. Divide a polynomial by a binomial.
3. Find the second root of a quadratic equation given one root.
4. * Find the complete solution set of a cubic equation given one root. (Optional)

VOCABULARY: polynomial, monomial, cubic equation, divisor, dividend, quotient

MOTIVATION:

If one factor of the expression $2x^3-6x^2+8x$ is 2x, what is the other factor.

DEVELOPMENT:

1. Use the motivation to review division of a polynomial by a monomial. Present $\frac{20k^2m^3-10m^2k^2-5m^4k+15m^2k^5}{5m^2k}$ and elicit the solution.

   (solution is $4km-2k-m^2+3k^3$) Elicit that you may only divide by a factor if it is present in all terms.

   Then present $\frac{12y^3z^2-9y^4z-5z^2+12y^3z}{4y^2z}$.

   The solution is $\frac{12y^3z-9y^4-5z+12y^3}{4y^2}$ since the coefficients are all not divisible by the same number and since y does not appear in each term.

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2. Present $(x^3+x^2-x-1)\div(x-1)$. Elicit that this can't be done term by term. This is a long division problem much like $7625\div33$. To find the quotient for the arithmetic example we would do $33\overline{)7625}$. Elicit the steps, one at a time and do the original example side by side. That is

$$\begin{array}{r} 2 \\ 33\overline{)7625} \\ \underline{66\downarrow} \\ 102 \end{array}$$

$$\begin{array}{rl} & x^2 \\ x-1 & \overline{)x^3+x^2-x-1} \\ & \underline{-x^3\overset{+}{\ominus}x^2\downarrow} \\ & 2x^2-x \end{array}$$

Elicit that we must subtract just as we did in the first example. That is we must change signs and add.

Ask for the next step in each and then you should be able to elicit the complete solutions.

That is we divide again.

$$\begin{array}{r} 231 \text{ R2 or } \frac{2}{33} \\ 33\overline{)7625} \\ \underline{66\downarrow\downarrow} \\ 102 \\ \underline{99}\downarrow \\ 35 \\ \underline{33} \\ 2 \end{array}$$

$$\begin{array}{rl} & x^2+2x+1 \\ x-1 & \overline{)x^3+x^2-x-1} \\ & \underline{x^3\overset{+}{\ominus}x^2\downarrow\downarrow} \\ & 2x^2-x \\ & \underline{2x^2\overset{+}{\ominus}2x}\downarrow \\ & x-1 \\ & \underline{x\overset{+}{\ominus}1} \\ & 0 \end{array}$$

Ask - could the division of a polynomial by a binomial yield a remainder? Yes - if the last coefficient in this problem was -3 there would be a remainder of -2.

3. Present $(4y^3-y+16)\div(2y+3)$. Ask what is different about the dividend in this example? (It is incomplete.) What is missing? (a y^2 term). In order to do this division properly we should complete the polynomial as $4y^3+0y^2-y+16$ and divide as above.

$$\begin{array}{r r}
 & 2y^2 \\
2y+3\,\big) & \overline{4y^3+0y^2-y+16} \\
 & \underline{4y^3\overset{-}{\oplus}6y^2} \\
 & -6y^2-y
\end{array}$$

Be sure to indicate that we must still change signs before subtracting and not just bring down the $6y^2$. Hence we get $-6y^2$.

Then completely elicit the solution as

$$\begin{array}{r r}
 & 2y^2-3y+4 \\
2y+3\,\big) & \overline{4y^3+0y^2-y+16} \\
 & \underline{4y^3\overset{-}{\oplus}6y^2} \\
 & -6y^2-y \\
 & \underline{-6y^2\overset{+}{\ominus}9y} \\
 & 8y+16 \\
 & \underline{8y\overset{-}{\oplus}12} \\
 & 4
\end{array}
\quad R_4 \text{ or } 2y^2-3y+4\frac{4}{2y+3}$$

Present the equation $2x^2-14x+24=0$. Given that one root is 4, what is the other roots. Elicit that since a root is 4, then a factor is $x-4$. To find the other root we can write $(x-4)(\quad) = 2x^2-14x+24$ and get the other factor as $2x-6$ which yields 3 as the other root OR we could divide

$$x-4\,\big)\overline{2x^2-14x+24}.$$

Now lets consider the equation $x^3-4x^2+x+6=0$ with one root $= 2$. How could we find the other roots? The only way is to divide

$$\begin{array}{r r}
 & x^2-2x-3 \\
x-2\,\big) & \overline{x^3-4x^2+x+6} \\
 & \underline{x^3\overset{+}{\ominus}2x^2} \\
 & -2x^2+x \\
 & \underline{-2x^2\overset{-}{\oplus}4x} \\
 & -3x+6 \\
 & \underline{-3x\overset{-}{\oplus}6} \\
 & 0
\end{array}$$

and get $x^2-2x-3=0$ and by factoring into $(x-3)(x+1) = 0$ we get the other roots are 3, -1.

4. (Optional) What are the roots of $x^3-6x^2+11x-6=0$? In order to solve this we could use the previous method if we knew one root. Elicit that at least one of the roots must be a factor of 6. Therefore we could try $\underline{+}1$, $\underline{+}2$, $\underline{+}3$, $\underline{+}6$ to achieve one root. Have each row test one pair of roots to establish that +2 is a root and thus x-2 is a factor. From here we could continue to test for other roots or divide

$$
\begin{array}{r}
x^2-4x+3 \\
x-2\overline{)\,x^3-6x^2+11x-6} \\
\underline{x^3 \overset{+}{\ominus} 2x^2 \quad\quad\quad} \\
-4x^2+11x \quad \\
\underline{-4x^2 \overset{-}{+} 8x \quad} \\
3x-6 \\
\underline{3x \overset{+}{-} 6}
\end{array}
$$

and get $x^2-4x+3=0$

$(x-3)(x-1)=0$

$x=3$, $x=1$, and $x=2$

are the roots.

APPLICATIONS:

1. Divide $\dfrac{12x^2y+xy^2-7xy^2-x^2y^3}{xy}$

2. Divide $a^3-7a^2+10a+8$ by $a-4$

3. Divide $6n^2-4+4n^3$ by $n+2$

4. Divide $4x^3-6x^2y+4xy^2-2y^3$ by $x-y$

* 5. Find the complete solution set of $6m^3-17m^2-5m+6=0$ if one root is 3.

* 6. Find the solution set for $x^3+4x^2-x-4=0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\dfrac{12x^2y+xy^2-7x^2y-x^2y^3}{xy} = 12x+y-7x-xy^2 = 5x+y-xy^2$

2.
$$\begin{array}{rr} & a^2-3a+22 \qquad \frac{96}{a-4} \\ a-4 \big) & \overline{a^3-7a^2+10a+8} \\ & \underline{a^3 \overset{+}{\ominus} 4a^2 \qquad\qquad} \\ & -3a^2+10a \qquad \\ & \underline{-3a^2 \overset{+}{\ominus} 12a \qquad} \\ & 22a+8 \\ & \underline{22a \overset{+}{\ominus} 88} \\ & 96 \end{array}$$

3.
$$\begin{array}{rr} & 4n^2-2n+4 \qquad \frac{-12}{n+2} \\ n+2 \big) & \overline{4n^3+6n^2+0n-4} \\ & \underline{4n^3 \overset{-}{\oplus} 8n^2 \qquad\qquad} \\ & -2n^2+0n \qquad \\ & \underline{-2n^2 \overset{+}{\ominus} 4n \qquad} \\ & 4n-4 \\ & \underline{4n \overset{-}{\oplus} 8} \\ & -12 \end{array}$$

4.
$$\begin{array}{rr} & 4x^2-2xy+2y^2 \\ x-y \big) & \overline{4x^3-6x^2y+4xy^2-2y^3} \\ & \underline{4x^3 \overset{+}{\ominus} 4x^2y \qquad\qquad\qquad} \\ & -2x^2y+4xy^2 \qquad \\ & \underline{-2x^2y \overset{-}{\oplus} 2xy^2 \qquad} \\ & 2xy^2-2y^3 \\ & \underline{2xy^2 \overset{+}{\ominus} 2y^3} \\ & 0 \end{array}$$

* 5.

$$\begin{array}{r} 6m^2+m-2 \\ m-3\overline{)6m^3-17m^2-5m+6} \\ \underline{6m^3\overset{+}{\ominus}18m^2} \\ m^2-5m \\ \underline{m^2\overset{+}{\ominus}3m} \\ -2m+6 \\ \underline{-2m\overset{-}{\oplus}6} \end{array}$$

$6m^2+m-2=0$

$(2m-1)(3m+2)=0$

$2m-1=0$ or $3m+2=0$

$m=1/2 \qquad m=-2/3$

Complete solution set = $\left\{3,\ 1/2,\ -2/3\right\}$.

* 6. For $x^3+4x^2-x-4=0$ try -1 as a root, therefore $x+1$ is a factor.

$$\begin{array}{r} x^2+3x-4 \\ x+1\overline{)x^3+4x^2-x-4} \\ \underline{x^3\overset{-}{\oplus}x^2} \\ 3x^2-x \\ \underline{3x^2\overset{-}{\oplus}3x} \\ -4x-4 \\ \underline{-4x\overset{+}{\ominus}4} \\ 0 \end{array}$$

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x=-4, \quad x=+1$

Complete solution set is $\left\{\pm1,\ -4\right\}$.

SUMMARY:

1. How do we divide a polynomial by a binomial?
2. How do we find all roots of an equation given one root?

LESSON: 14

AIM: How do we multiply and divide rational expressions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Simplify rational expressions by factoring.
2. Multiply and simplify rational expressions.
3. Divide and simplify rational expressions.

* 4. Simplify rational expressions involving cubic expressions.

VOCABULARY: factor, rational expression, invert, reciprocal

MOTIVATION:

Simplify each of the following:

a) $\frac{20}{35}$

b) $\frac{3x^2y}{x}$

c) $\frac{x^2+3x+2}{x^2-3x-10}$

DEVELOPMENT:

1. Elicit the solutions to the motivating examples: (a) $\frac{4}{7}$ (b) $3xy$ (c) $\frac{(x+1)\cancel{(x+2)}}{\cancel{(x+2)}(x-5)} = \frac{x+1}{x-5}$

   Now present $\frac{12x}{5b} \cdot \frac{b^2}{x}$ Elicit that we could simplify by dividing by common factors as $\frac{12\overset{1}{\cancel{x}}}{5\underset{1}{\cancel{b}}} \cdot \frac{b^{\cancel{2}}}{\underset{1}{\cancel{x}}} = \frac{12b}{5}$

   Have students simplify $\frac{3a}{10} \cdot \frac{5a-5b}{a^2+4ab}$ Elicit that in order to divide by common factors we must first factor the fraction at the right. That is $\frac{3a}{10} \cdot \frac{5(a-b)}{a(a+4b)}$ then dividing we get $\frac{3\cancel{a}}{\cancel{10}^{2}} \cdot \frac{\cancel{5}(a-b)}{\cancel{a}(a+4b)} = \frac{3(a-b)}{2(a+4b)}$

Now lets try a more involved problem.

Find the product in simplest form:

$$\frac{w^2+3w-18}{w^2-16} \cdot \frac{w^2-6w-40}{w^2+2w-24}$$

Again elicit that we must factor all the expressions before dividing

$$\frac{(w+6)(w-3)}{(w+4)(w-4)} \cdot \frac{(w-4)(w+10)}{(w+6)(w-4)}$$

Therefore, dividing by common factors elicit

$$\frac{\overset{1}{\cancel{(w+6)}}(w-3)}{(w+4)\underset{1}{\cancel{(w-4)}}} \cdot \frac{\overset{1}{\cancel{(w-4)}}(w+10)}{\underset{1}{\cancel{(w+6)}}(w-4)} = \frac{(w-3)(w+10)}{(w+4)(w-4)}$$

2. How do we divide fractions such as $\frac{3}{5} \div \frac{9}{20}$? Elicit we invert or take the reciprocal of the divisor and get $\frac{3}{5} \cdot \frac{20}{9} = \frac{\overset{1}{\cancel{3}}}{\cancel{5}_{1}} \cdot \frac{\cancel{20}^{4}}{\cancel{9}_{3}} = \frac{4}{3}$

   Ask class to divide $\frac{a^2-ab}{ab+2b^2} \div \frac{a^2+ab}{ab+b^2}$

   Elicit first we must rewrite the problem as a multiplication example using the reciprocal of the divisor.

   $\frac{a^2-ab}{ab+2b^2} \cdot \frac{ab+b^2}{a^2+ab}$ Elicit need to factor.

   $\frac{a(a-b)}{b(a+2b)} \cdot \frac{b(a+b)}{a(a+b)}$ Elicit need to divide by common factors.

   $\frac{\overset{1}{\cancel{a}}(a-b)}{\underset{1}{\cancel{b}}(a+2b)} \cdot \frac{\overset{1}{\cancel{b}}\overset{1}{\cancel{(a+b)}}}{\underset{1}{\cancel{a}}\underset{1}{\cancel{(a+b)}}} = \frac{a-b}{a+2b}$ Discuss why neither the a nor b can be divided.

3. * (Optional) Consider the fraction

   $\frac{x^3+2x^2-11x-12}{x^2-4x-5}$ Elicit need to factor to simplify. Since the denominator is easily factored we get $\frac{x^3+2x^2-11x-12}{(x-5)(x+1)}$

Elicit this gives as a hint that (x+1) must be a factor of the numerator if we are to simplify, so we divide

$$\begin{array}{r} x^2+x-12 \\ x+1\overline{\big)\,x^3+2x^2-11x-12} \\ \underline{x^3\overset{-}{\oplus}x^2 \downarrow \quad\;\; \downarrow} \\ x^2-11x \quad\;\; \downarrow \\ \underline{x^2\overset{-}{\oplus}x \quad\;\;\; \downarrow} \\ -12x-12 \\ \underline{-12x\overset{+}{\ominus}12} \end{array}$$

Therefore we have

$$\frac{\overset{1}{\cancel{(x+1)}}(x^2+x-12)}{\underset{1}{\cancel{(x+1)}}(x-5)} = \frac{(x^2+x-12)}{(x-5)} \text{ or } \frac{(x+4)(x-3)}{(x-5)}$$

and we see that there are no other common factors.

APPLICATIONS:

Simplify each of the following.

1. $\dfrac{4a^2b^4}{8a^3b^7}$
2. $\dfrac{3x^2}{2x^2+3x}$
3. $\dfrac{y^2-3y}{4y-12}$
4. $\dfrac{m^2+4m}{m^2+6m+9} \cdot \dfrac{-m-3}{8m+32}$
5. $\dfrac{8a^2+2ab-15b^2}{12a^2-7ab-10b^2} \cdot \dfrac{36a^2-b^2}{12a^2+20ab+3b^2}$
6. $\dfrac{4x-8}{x^3+3x^2} \div \dfrac{6x-12}{x^2+3x}$
7. $\dfrac{x^2-3x+2}{7x-7x^2} \div \dfrac{8-6x+x^2}{14x^3-56x^2}$

* 8. $\dfrac{p^4-15p^2-10p+24}{p^3-p^2-12p}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\dfrac{{}^{1}\cancel{4}\cancel{a}^{\cancel{2}}\cancel{b}^{\cancel{4}}}{{}_{2}\cancel{8}a^{\cancel{3}}b^{\cancel{7}3}} = \dfrac{1}{2ab^3}$

2. $\dfrac{3x^2}{2x^2+3x} = \dfrac{3x^{\cancel{2}}}{\cancel{x}(2x+3)} = \dfrac{3x}{2x+3}$

3. $\dfrac{y\ \ -3y}{4y-12} = \dfrac{y\overset{1}{\cancel{(y-3)}}}{4\underset{1}{\cancel{(y-3)}}} = \dfrac{y}{4}$

4. $\dfrac{m^2+4m}{m^2+6m+9} \cdot \dfrac{-m-3}{8m+32} = \dfrac{m(m+4)}{(m+3)(m+3)} \cdot \dfrac{-1(m+3)}{8(m+4)} = \dfrac{-1\overset{1}{\cancel{(m+3)}}}{(m+3)\underset{1}{\cancel{(m+3)}}}$

$= \dfrac{-1}{(m+3)}$

5. $\dfrac{8a^2+2ab-15b^2}{12a^2-7ab-10b^2} \cdot \dfrac{36a^2-b^2}{12a^2+20ab+3b^2} =$

$\dfrac{(4a-5b)(2a+3b)}{(4a-5b)(3a+2b)} \cdot \dfrac{(6a-b\ )\overset{1}{\cancel{(6a+b)}}}{(2a+3b)\underset{1}{\cancel{(6a+b)}}} = \dfrac{6a-b}{3a+2b}$

6. $\dfrac{4x-8}{x^3+3x^2} \div \dfrac{6x-12}{x^2+3x} = \dfrac{{}^{2}\cancel{4}\overset{1}{\cancel{(x-2)}}}{x^{\cancel{2}}\underset{1}{\cancel{(x+3)}}} \cdot \dfrac{\cancel{x}\overset{1}{\cancel{(x+3)}}}{{}_{3}\cancel{6}\underset{1}{\cancel{(x-2)}}} = \dfrac{2}{3x}$

7. $\dfrac{x^2-3x+2}{7x-7x^2} \div \dfrac{8-6x+x^2}{14x^3-56x^2} = \dfrac{\overset{1}{\cancel{(x-2)}}\overset{1}{\cancel{(x-1)}}}{(7x)(-1)\underset{1}{\cancel{(x-1)}}} \cdot \dfrac{(14x^2)\overset{1}{\cancel{(x-4)}}}{\underset{1}{\cancel{(x-2)}}\underset{1}{\cancel{(x-4)}}}$

$= \dfrac{(14x^2)}{(7x)(-1)} = \dfrac{2x}{-1} = -2x$

8. $\dfrac{p^4-15p^2-10p+24}{p^3-p^{2-12p}} = \dfrac{(p+3)(p^3-3p^2-6p+8)}{p(p-4)(p+3)} = \dfrac{\overset{1}{\cancel{(p+3)}}\overset{1}{\cancel{(p-4)}}(p^2+p-2)}{p\underset{1}{\cancel{(p-4)}}\underset{1}{\cancel{(p+3)}}}$

$= \dfrac{p^2+p-2}{p}$ or $\dfrac{(p+2)(p+1)}{p}$

SUMMARY:

How do we divide (multiply) algebraic fractions?

LESSON: 15

AIM: How do we add and subtract rational expressions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Add rational expressions.
2. Subtract rational expressions.

VOCABULARY: addition, subtraction, rational expression, least common denominator (L.C.D)

MOTIVATION:

If Tom can do 2/3 of a job in an hour and Bill can do 1/4 of the same job in an hour, how much of the job will they complete in an hour.

DEVELOPMENT:

1. Elicit that we must add $\frac{2}{3} + \frac{1}{4}$ and in order to do this we need a common denominator (L.C.D). That is

$$\begin{array}{rcl} & \frac{2}{3} = & \frac{8}{12} \\ + & \frac{1}{4} = & \frac{3}{12} \\ \hline & & \frac{11}{12} \end{array}$$

Suppose we wanted to add $\frac{2}{5x} + \frac{3}{10x}$.

What would we need? (An L.C.D)

What is the L.C.D. of 5x and 10x? (10x)

Therefore we want to change the fractions to equivalent fractions with 10x as the common denominator. That is

$$\begin{array}{rcl} & \frac{2}{5x} = & \frac{\quad}{10x} \\ + & & \\ & \frac{3}{10x} = & \frac{3}{10x} \\ \hline \end{array}$$

What will be the numerator of the first fraction?

Elicit since we multiply 5x by 2 to get 10x. Therefore we must multiply 2 by 2 and get $\frac{4}{10x} + \frac{3}{10x} = \frac{7}{10x}$

How do we add and subtract arithmetic fractions?

Elicit when adding or subtracting arithmetic fractions, we combine the numerators and keep the common denominator. Ask: How can we add or subtract rational expressions? Elicit when adding (or subtracting) rational expressions we combine the numerators and keep the common denominator.

2. Ask class to simplify $\frac{2}{y} - \frac{1}{y^2} + \frac{3}{2y}$.

What must we get? (A common denominator or preferably the L.C.D.)

Elicit that to do this we want an expression that all of the denominators will be able to be divided into evenly.

Elicit that $2y^2$ is the L.C.D.

so $\frac{2}{y} - \frac{1}{y^2} + \frac{3}{2y}$ must be rewritten

as $\frac{2 \cdot 2y}{2y^2} - \frac{2 \cdot 1}{2y^2} + \frac{y \cdot 3}{2y^2}$.

Elicit procedure needed and

$$\frac{4y-2+3y}{2y^2} = \frac{7y-2}{2y^2}.$$

Can we reduce this fraction? (No since all three terms do not have a common factor.)

Have class combine $\frac{3a}{a-1} - \frac{4}{a-2}$. Elicit since the denominators are relatively prime (have no common factors), the L.C.D is (a-1)(a-2) so we have

$\frac{\quad}{(a-1)(a-2)} - \frac{\quad}{(a-1)(a-2)}$ as the denominators.

To get the new numerators we must multiply $3a \cdot (a-2)$ and 4 by a-1 to get $\frac{3a(a-2)}{(a-2)(a-1)} - \frac{4(a-1)}{(a-2)(a-1)}$ we now must

multiply to get $\frac{3a^2-6a-(4a-4)}{(a-2)(a-1)}$.

Note: That since we are subtracting we must change all the signs after the subtraction sign and we get

$$\frac{3a^2-6a-4a+4}{(a-2)(a+1)}.$$

Combining we get $\frac{3a^2-10a+4}{(a-2)(a-1)}$. Can this be simplified?

Elicit we must see if the numerator and denominator have common factors. Since there are no common factors, this is our final answer.

3. Now consider $\frac{2x}{x^2-2x-15} + \frac{3}{4x^2+12x}$. Do the denominators have any common factors? In order to find out what must we do? (factor each denominator). That is we have $\frac{2x}{(x-5)(x+3)} + \frac{3}{4x(x+3)}$. We now see that the L.C.D is $(x-5)(x+3)(4x)$. Therefore the first fraction must be multiplied by $\frac{4x}{4x}$ and the second fraction by $\frac{(x-5)}{(x-5)}$ which yields the following example.

$$\frac{2x(4x)}{4x(x-5)(x+3)} + \frac{3(x-5)}{4x(x-5)(x+3)} = \frac{8x^2+3x-15}{4x(x-5)(x+3)}$$

How can we determine if the fraction can be simplified? Elicit we must determine if the numerator contains any of the factors in the denominator. Have students find that it cannot be done. Therefore we have our final answer.

APPLICATIONS:

Simplify each of the following.

1. $\frac{7}{b^2} + \frac{3}{2b}$
2. $\frac{z}{z+3} - \frac{4}{z}$
3. $\frac{8}{y} + \frac{2}{3} - \frac{2}{y+4}$
4. $\frac{3x}{x-2} + \frac{x}{2x-1}$
5. $\frac{x}{x-4} + \frac{6}{x} + \frac{2x+8}{x^2-4x}$
6. $\frac{a}{a^2-9a+18} - \frac{a-2}{a^2-10a+24}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\frac{7}{b^2} + \frac{3}{2b} = \frac{14}{2b^2} + \frac{3b}{2b^2} = \frac{14+3b}{2b^2}$

2. $\frac{z}{z+3} - \frac{4}{z} = \frac{z^2-4z-12}{z(z+3)}$ or $\frac{(z-6)(z+2)}{z(z+3)}$

3. $\frac{8}{y} + \frac{2}{3} - \frac{2}{y+4} = \frac{8(3)(y+4)+2(y)(y+4)-2(3y)}{3y(y+4)}$

   $= \frac{24y+96+2y^2+8y-6y}{3y(y+4)} = \frac{2y^2+26y+96}{3y(y+4)}$

4. $\frac{3x}{x-2} + \frac{x}{2x-1} = \frac{3x(2x-1)+x(x-2)}{(x-2)(2x-1)} = \frac{6x^2-3x+x^2-2x}{(x-2)(2x-1)} = \frac{7x^2-5x}{(x-2)(2x-1)}$

   $= \frac{x(7x-5)}{(x-2)(2x-1)}$

5. $\frac{x}{x-4} + \frac{6}{x} + \frac{2x+8}{x^2-4x} = \frac{x(x)+6(x-4)+2x+8}{x^2-4x} = \frac{x^2+6x-24+2x+8}{x^2-4x}$

   $= \frac{x^2+8x-16}{x(x-4)}$

6. $\frac{a}{a^2-9a+18} - \frac{a-2}{a^2-10a+24} = \frac{a}{(a-3)(a-6)} - \frac{a-2}{(a-4)(a-6)}$

   $= \frac{a(a-4)-(a-2)(a-3)}{(a-3)(a-4)(a-6)} = \frac{a^2-4a-(a^2-5a+6)}{(a-3)(a-4)(a-6)} = \frac{a^2-4a-a^2+5a-6}{(a-3)(a-4)(a-6)}$

   $\frac{\overset{1}{\cancel{a-6}}}{(a-3)(a-4)\underset{1}{\cancel{(a-6)}}} = \frac{1}{(a-3)(a-4)}$

SUMMARY:

How do we combine algebraic fractions with

a) A variable in the denominator such as $\frac{3}{x} + \frac{4}{2x}$?

b) Polynomial expressions such as $\frac{x-7}{x^2+4x-5} - \frac{x-9}{x^2+3x-10}$?

LESSON: 16

AIM: How do we simplify complex fractions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Simplify complex fractions containing rational numbers.
2. Simplify complex fractions containing monomials.

VOCABULARY: complex fraction, divisor, least common denominator (L.C.D.), monomial

MOTIVATION:

My son had three quarters (3/4) of a candy bar that I insisted he share with his brother so he had to break it in 1/2. How much did he give his brother.

DEVELOPMENT:

1. Elicit that we must represent this mathematically. Lets try a simpler problem. Suppose he had a whole candy bar. We could represent it in two ways. What are they? (1÷2 or $\frac{1}{2}$) Let us now consider the original example. We begin with 3/4 so we could have $\frac{3}{4} \div 2$ or $\frac{\frac{3}{4}}{2}$. Elicit that this is something new, a fraction within a fraction. Explain this is called a complex fraction. (The aim can now be elicited.) In the first case we get
   $\frac{3}{4} \div \frac{2}{1} = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$
   What should the answer be for the other representation? ($\frac{3}{8}$). In what other way could we simplify this complex fraction?
   Elicit: $\frac{\frac{3}{4}}{2} \cdot \frac{4}{4}$. The numerator becomes $\frac{3}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = \frac{3}{1} = 3$
   The denominator $2 \cdot 4 = 8$, therefore $\frac{\frac{3}{4}}{2} \cdot \frac{4}{4} = \frac{3}{8}$

Lets try to simplify $\dfrac{\frac{4}{7}}{\frac{2}{21}}$.

Elicit Method 1: $\dfrac{4}{7} \div \dfrac{2}{21} = \dfrac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{7}}} \cdot \dfrac{\overset{3}{\cancel{21}}}{\underset{1}{\cancel{2}}} = \dfrac{6}{1} = 6$

Elicit Method 2: Have class suggest finding a number that we can multiply both numerator and denominator by to eliminate the complex fraction. This is the L.C.D. (Least Common Denominator). For this example the L.C.D. is 21. That is

$$\frac{\frac{4}{7}}{\frac{2}{21}} \cdot \frac{21}{21} = \frac{\frac{4}{\cancel{7}}}{\frac{2}{\underset{1}{\cancel{21}}}} \cdot \frac{\overset{3}{\cancel{21}}}{\underset{1}{\cancel{21}}} = \frac{12}{2} = \frac{6}{1} = 6$$

2. Lets try one involving variables. Simplify

$$\frac{\frac{3}{2} + \frac{3}{x}}{4 + \frac{4}{x}}.$$

Elicit using method 1 we would have to get common denominators in both the numerator and denominator, add these new fractions, then divide. While this could be done, (see below), it is not the best method. Therefore using method 2, we need to find the L.C.D. of the 3 denominators in the complex fraction? (2x) Therefore by method 2 we get

$$\frac{\left(\frac{3}{2} + \frac{3}{x}\right)}{\left(4 + \frac{4}{x}\right)} \cdot \frac{2x}{2x}$$ Emphasize the need to

use the distributive property and get

$$\frac{\frac{3}{2} \cdot 2x + \frac{3}{x} \cdot 2x}{4 \cdot 2x + \frac{4}{x} \cdot 2x} = \frac{3x+6}{8x+4} = \frac{3\overset{1}{\cancel{(x+2)}}}{4\underset{1}{\cancel{(x+2)}}} = \frac{3}{4}$$

Elicit using the other method, we need to combine the

$$\frac{\frac{3}{2} + \frac{3}{x}}{4 + \frac{4}{x}}$$

fractions in the numerators into a single term and the fractions in the denominator into a single term.

Elicit $\dfrac{\frac{3}{2}+\frac{3}{x}}{4+\frac{4}{x}} = \dfrac{\frac{3x}{2x}+\frac{6}{2x}}{\frac{4x}{x}+\frac{4}{x}} = \dfrac{\frac{3x+6}{2x}}{\frac{4x+4}{x}}$

$$= \frac{3x+6}{2x}\cdot\frac{x}{4x+4} = \frac{3x+6}{2(4x+4)} = \frac{3\overset{1}{\cancel{(x+2)}}}{4\underset{1}{\cancel{(x+2)}}} = \frac{3}{4}$$

Have students see both methods yield the same result.

* 3. Optional for Honors classes.

Simplify $\dfrac{\frac{1}{z-1}-\frac{1}{z+2}}{1+\frac{1}{z^2+z-2}}$ Have class determine the L.C.D. is $(z+2)(z-1)$

is z^2-z-2 and multiply each term by this.

Note: $z^2-z-2 = (z+2)(z-1)$ hense we get

$$\frac{(z+2)(z-1)\cdot\frac{1}{(z-1)} - \frac{1}{(z+2)}\cdot(z+2)(z-1)}{(z+2)(z-1)+1}$$

$$= \frac{(z+2)(z-1)}{(z+2)(z-1)+1} = \frac{(z+2)(z-1)}{z^2-z+1}$$

This new denominator is not factorable, hense this is the final answer.

APPLICATIONS:

Simplify each complex fraction.

1. $\dfrac{\frac{3}{4}-\frac{5}{8}}{\frac{3}{8}}$

2. $\dfrac{\frac{r}{2}+\frac{s}{3}}{\frac{r}{4}-\frac{s}{8}}$

3. $\dfrac{y-\frac{1}{y}}{2-\frac{2}{y}}$

4. $\dfrac{\frac{7}{3}+\frac{1}{x}}{\frac{2}{x}-\frac{1}{3}}$

5. $$\frac{\frac{2}{y}-\frac{1}{2z}}{\frac{3}{y}-\frac{1}{3z}}$$

6. $$\frac{y^{-1}+2}{x+1}$$

* 7. (Optional) $$\frac{1-\frac{6}{x^2+x}}{1-\frac{8}{x+1}+\frac{10}{x^2+x}}$$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $$\frac{\left(\frac{3}{4}-\frac{5}{8}\right)}{\left(\frac{3}{8}\right)}\cdot\frac{8}{8}=\frac{6-5}{3}=\frac{1}{3}$$

2. $$\frac{\left(\frac{r}{2}+\frac{s}{3}\right)}{\left(\frac{r}{4}-\frac{s}{8}\right)}\cdot\frac{24}{24}=\frac{12r+8s}{6r-3s}=\frac{4(3r+2s)}{3(2r-s)}$$

3. $$\frac{\left(y-\frac{1}{y}\right)}{\left(2-\frac{2}{y}\right)}\cdot\frac{y}{y}=\frac{y^2-1}{2y-2}=\frac{(y+1)(y-1)}{2(y-1)}=\frac{y+1}{2}$$

4. $$\frac{\left(\frac{7}{3}+\frac{1}{x}\right)}{\left(\frac{2}{x}-\frac{1}{3}\right)}\cdot\frac{3x}{3x}=\frac{7x+3}{6-x}$$

5. $$\frac{\left(\frac{2}{y}-\frac{1}{2z}\right)}{\left(\frac{3}{y}-\frac{1}{3z}\right)}\cdot\frac{6yz}{6yz}=\frac{12z-3y}{18z-2y}=\frac{3(4z-y)}{2(9z-y)}$$

6. $$\frac{y^{-1}+2}{x+1}=\frac{\left(\frac{1}{y}+2\right)}{(x+1)}\cdot\frac{y}{y}=\frac{1+2y}{y(x+1)}$$

7. $$\frac{\left(1-\frac{6}{x^2+x}\right)}{\left(1-\frac{8}{x+1}+\frac{10}{x^2+x}\right)}\cdot\frac{x^2+x}{x^2+x}=\frac{x^2+x-6}{x^2+x-8x+10}=\frac{(x+3)(x-2)}{x^2-7x+10}$$

$$=\frac{(x+3)\overset{1}{\cancel{(x-2)}}}{(x-5)\underset{1}{\cancel{(x-2)}}}=\frac{x+3}{x-5}$$

SUMMARY:

Explain the procedures that may be used to simplify a complex fraction?

LESSON: 17

AIM: How to solve fractional equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve proportions.
2. Solve fractional equations.

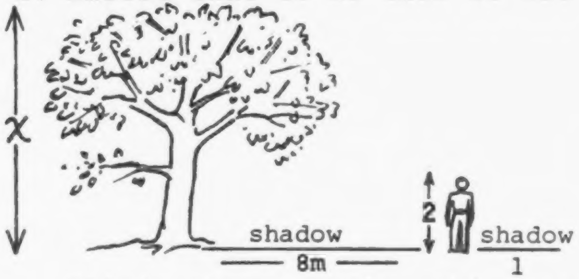
VOCABULARY: proportion, fractional equations, L.C.D. least common denominator, means, extremes

MOTIVATION:

How tall is a tree that casts a shadow of 8m at the same time a man 2m tall casts a shadow of 1m?

DEVELOPMENT:

1. Elicit that to do this we set up a proportion.



$$\frac{x}{8} = \frac{2}{1}$$

Elicit that to solve this proportion we multiply the means and set it equal to the product of the extremes to get $x = 16$.

Present $\frac{6}{x+2} = \frac{7}{8}$. Multiplying we get $7(x+2) = 48$ which gives us $7x+14 = 48$ then solve to get $7x = 34$ and $x = \frac{34}{7}$

Lets consider $\frac{2}{y} = \frac{y-1}{3}$ Elicit that cross multiplying we get $y^2-y = 6$. We must re-write this in standard form and solve. $y^2-y-6 = 0$

$(y-3)(y+2) = 0$

$y = 3$ or $y = -2$

2. Now lets consider $\frac{4}{x} - \frac{1}{x} = 3$

What must we do in order to solve this? Elicit that we want to eliminate the denominators and that multiplying by x will accomplish this. Multiplying by x we get

$$\begin{aligned} 4-1 &= 3x \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

Present $\frac{1}{3} - \frac{2}{y} = \frac{1}{12}$. What is the L.C.D. of 3, y, 12?

Elicit 12y is the L.C.D. Therefore multiplying we get

$$\overset{4}{\cancel{12y}} \cdot \frac{1}{\underset{1}{\cancel{3}}} - 12\cancel{y} \cdot \frac{2}{\cancel{y}} = \frac{1}{\cancel{12}} \cdot \cancel{12}y$$

$4y-24=y$; $3y=24$; $y=8$

3. Lets try $\frac{3x^2}{x-1} + \frac{3}{2} = \frac{2x-1}{x-1}$ Elicit that $2(x-1)$ is the L.C.D.

Multiplying we get $2 \cdot 3x^2+3(x-1) = 2(2x-1)$. This gives us $6x^2+3x-3 = 4x-2$ Putting this in standard form we get

$6x^2-x-1 = 0$

$(3x+1)(2x-1) = 0$

$3x+1 = 0$ or $2x-1 = 0$

$x = -1/3 \quad x = 1/2$

APPLICATIONS:

Solve each of the following.

1. $\frac{m-1}{4} = \frac{5}{6}$
2. $\frac{1}{n} - \frac{1}{2} = 1$
3. $\frac{1}{x^2} + \frac{2}{x} = \frac{9}{16}$
4. $\frac{y}{y-2} + \frac{y}{y+1} = 2$
5. $\frac{3a}{a-2} - \frac{2a}{a-3} = \frac{a+27}{a^2-5a+6}$
6. $\frac{2x}{x+5} - \frac{1}{5-x} = \frac{10}{x^2-25}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\frac{m-1}{4} = \frac{5}{6}$ $\quad 6(m-1)=20 \; ; \; 6m-6=20$

$$6m = 26$$

$$m = \frac{26}{6} = \frac{13}{3}$$

2. $\left(\frac{1}{n} - \frac{1}{2} = 1\right)2n \; ; \; 2-n=2n \; ; \; 2=3n \; ; \; n = \frac{2}{3}$

3. $\left(\frac{1}{x^2} + \frac{2}{x} = \frac{9}{16}\right)16x^2$

$$16+32x=9x^2 \; ; \; 9x^2-32x-16=0$$

$$(9x+4)(x-4) = 0$$

$$9x+4 = 0 \text{ or } x-4 = 0$$

$$9x = -4 \qquad x = 4$$

$$x = \frac{-4}{9}$$

4. $\left(\frac{y}{y-2} + \frac{y}{y+1} = 2\right)(y-2)(y+1)$

$$y(y+1)+y(y-2) = 2(y-2)(y+1)$$

$$y^2+y+y^2-2y = 2(y^2-y-2)$$

$$2y^2-y = 2y^2-2y-4$$

$$y = -4$$

5. $\left(\frac{3a}{a-2} - \frac{2a}{a-3} = \frac{a+27}{a^2-5a+6}\right)a^2-5a+6$

$$3a(a-3) - 2a(a-2) = a+27$$

$$3a^2 - 9a - 2a^2+4a = a+27$$

$$a^2-5a = a+27 \quad ; \; a^2-6a-27 = 0$$

$$(a-9)(a+3) = 0$$

$$a = 9 \text{ or } a = -3$$

6. $\frac{2x}{x+5} - \frac{1}{5-x} = \frac{10}{x^2-25}$

$$\left(\frac{2x}{x+5} + \frac{1}{x-5} = \frac{10}{x^2-25}\right)x^2-25$$

Note: $(5-x) = -(x-5)$

$$2x(x-5)+(x+5) = 10$$

$$2x^2-10x+x+5 = 10$$

$$2x^2-9x-5 = 0$$

$$(2x+1)(x-5) = 0$$

$$2x = -1 \text{ or } x = 5$$

$x = \frac{-1}{2}$ Impossible

SUMMARY:

How do we solve equations involving fractions?

LESSON: 18

AIM: How do we solve quadratic inequalities by factoring?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine when the factors of a product will be positive or negative (ie: if $ab>0$ and a is positive, what is b?)
2. Solve quadratic inequalities by factoring and graph the solution set on a number line.

VOCABULARY: factor, product, inequality

MOTIVATION:

Determine values for a and b so that each of the following will be true.

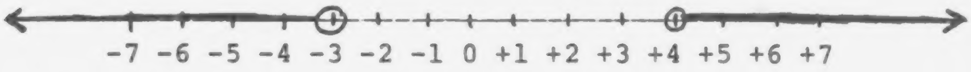
1) $a \cdot b = 0$
2) $a \cdot b > 0$
3) $a \cdot b < 0$

DEVELOPMENT:

1. Elicit that for $a \cdot b = 0$ either $a = 0$ or $b = 0$ or a $b = 0$. Review the logic symbols $\vee$ (or) and $\wedge$ (and) and their meanings.
   Ask is each of the following true or false.
   $3 > 4 \vee 4 < 6$ True since one statement is true.
   $3 > 4 \wedge 4 < 6$ False - Why? Elicit: For a conjunction to be true both statements must be true. With this in mind - lets consider $a \cdot b > 0$. Elicit that this means the product is positive. When is the product of two factor positive? When both are positive or both are negative. Symbolically, $a \cdot b > 0$ if $(a>0 \wedge b>0) \vee (a<0 \wedge b<0)$. Similarly $a \cdot b < 0$ if $(a>0 \wedge b<0) \vee (a<0 \wedge b>0)$ that is - if the product of two factors is negative, one factor must be positive and one negative. Write these statements on the board.

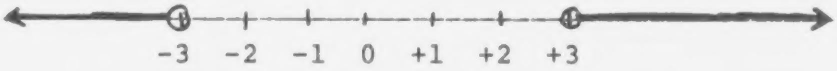
2. With this as a background. Present $a^2-a-12>0$. Elicit that it is a quadratic inequality. The term quadratic indicates that we should try to factor this in order to solve it. $a^2-a-12>0$; $(a-4)(a+3)>0$

   Elicit that now we have the situation described above $a\cdot b>0$. Therefore for $(a-4)(a+3)>0$ we have $(a-4>0 \wedge a+3>0) \vee (a-4<0 \wedge a+3<0)$. Therefore $(a>4 \wedge a>-3)$ $\vee (a<4 \wedge a<-3)$. Elicit that for $a>4 \wedge a>-3$ to be true both must be true therefore $a>4$. Similarly for $a<4 \wedge a<-3$ to be true $a<-3$. Therefore the solution is $a<-3 \vee a>4$. Lets show this on the number line.



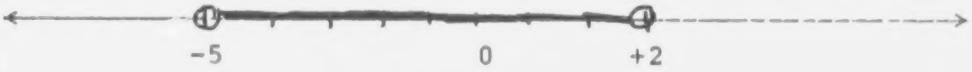
   Elicit that we use open circles to indicate ">" or "<".

   Now lets apply this to $x^2-9>0$. Factoring, we get $(a-3)(a+3)>0$. Therefore $(a-3>0 \wedge a+3>0) \vee (a-3<0 \wedge a+3<0)$ - thus $(a>3 \wedge a>-3)$ which gives us $a>3$. If $a<3 \wedge a<-3$, then $a<-3$. The complete solution is $a>3 \vee a<-3$. Elicit the graph.



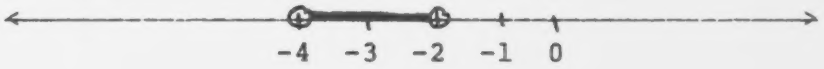
   Ask the class how this can be checked. Elicit we can check this by substitution. Have the students note that the solution to a "greater than" quadratic inequality will be two rays.

3. Now present $x^2+3x-10<0$. Elicit by factoring we get $(x+5)(x-2)<0$ and utilizing the concepts discussed previously involving $a\cdot b<0$ we get $(x+5<0 \wedge x-2>0)$ $\vee (x+5>0 \wedge x-2<0)$. That is $(x<-5) \wedge (x>2) \vee (x>-5) \wedge (x<2)$. Can the first statement ever be true? No, x cannot be less than -5 and greater than 2 so this case is rejected. Can $(x>-5) \wedge (x<2)$ be true? Yes - for values between -5 and 2. Ask class how this may be written. Elicit that this can be rewritten as $-5<x<2$, and graphed as



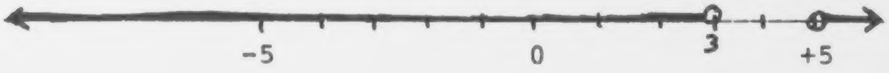
Elicit that we can check this by substitution. Now lets try $y^2+6y+8<0$. Elicit that factors $(y+2)(y+4)<0$. Therefore $(y+2>0 \wedge y+4<0) \vee (y+2<0 \wedge y+4>0)$. That is $(y>-2 \wedge y<-4) \vee (y<-2 \wedge y>-4)$ Which pair can be true?

Elicit the second pair. That is we can't find a number that is both greater than -2 and less than -4. So the solution is $-4<y<-2$ and the graph is



Elicit we can check by substituting a member of the solution set. (ie: -3, -2.5, etc.)

4. Solve $\frac{m-3}{m-5} > 0$ Ask: When will this fraction be undefined? When $m = 5$. Why? (The denominator will then be = 0 when $m = 5$.) For $m \neq 5$. When is a fraction positive? Elicit when numerator and denominator are the same sign. That is, analogously $\frac{a}{b} > 0$ iff$((a>0)\ (b>0))\ \ ((a<0)\ (b<0))$. Elicit that if $\frac{m-3}{m-5} > 0$ then $((m-3>0) \wedge (m-5>0)) \vee (m-3<0) \wedge (m-5<0))$. Therefore $m>3\ \ m>5$ which yields $m>5$ or $m<3 \wedge m<5$ which yields $m<3$. Therefore the solution is $m>5 \vee m<3$. Graphing this we get



5. Similarly ask class to state the conclusion if $\frac{a}{b} < 0$ and use this conclusion to solve $\frac{k}{k+6} < 0$.

APPLICATIONS:

Find the solution set and graph.

1. $x_2+6x+5>0$

2. $y^2>6y+16$

3. $x^2-7x+12<0$

4. $2n^2-8n-10<0$

5. $\frac{x+6}{x+3} > 0$

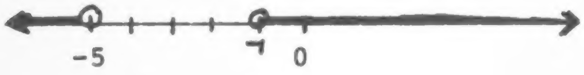
6. $\frac{y+5}{y-2} < 0$

* 7. $\frac{x^2+3x-4}{x-2} > 0$

SOLUTIONS TO APPLICATION PROBLEMS:

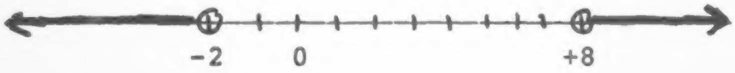
1. $x^2+6x+5>0$; $(x+5)(x+1)>0$ $\qquad x>-5 \wedge x>-1 \quad \therefore x>-1$

or $\qquad x<-5 \wedge x<-1 \quad \therefore x<-5$



$\{x<-5 \text{ or } x>-1\}$

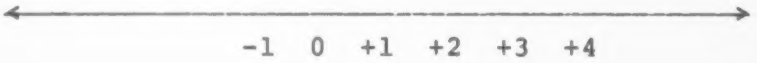
2. $y^2>6y+16$; $y^2-6y-16>0$; $(y-8)(y+2)>0$ Therefore $(y>8 \wedge y>-2)$. Therefore $y>8$. $y<8 \wedge y<-2$. Therefore $y<-2$.



solution set $\{y<-2 \text{ or } y>8\}$

3. $x^2-7x+12<0$; $(x-4)(x-3)<0$; $x<4$ $x>3$ or $x>4$ $x<3$ Reject

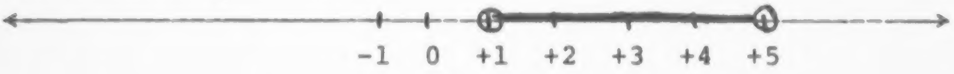
$\therefore 3<x<4$



4. $2n^2-8n-10<0$; $(2n+2)(n-5)<0$; $(2n+2>0)\wedge(n-5<0)$

$(2n+2<0)\wedge(n-5>0)$ $\therefore n>1 \wedge n<5$ solution or $n<1 \wedge n>5$,

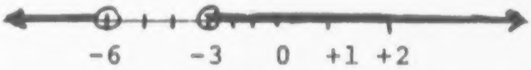
Reject $\therefore$ $\{1<n<5\}$ solution set



5. $\frac{x+6}{x+3} > 0$; $((x+6>0)\ (x+3>0)) \vee ((x+6<0)\wedge(x+3<0))$

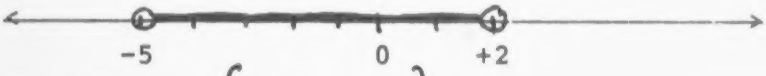
$x>-6 \wedge x>-3 \quad \therefore x>-3$; $x<-6 \wedge x<-3 \quad \therefore x<-6$

solution set $\{x<-6 \vee x>-3\}$



6. $\frac{y+5}{y-2} < 0$; $(y+5>0)\wedge(y-2<0)) \vee (y+5<0)\wedge(y-2>0)$

$y>-5 \wedge y<2$ or $y<-5 \wedge y>2$ Reject



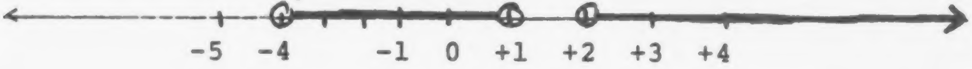
solution set $\{y>-5 \wedge y<2\}$

* 7. $\frac{x^2+3x-4}{x-2} < 0$; $\left[(x^2+3x-4>0)(x-2>0)\right] \vee \left[x^2+3x-4<0)(x-2<0)\right]$

$x^2+3x-4>0$ means $(x+4)(x-1)>0$. That is $(x+4)>0 \wedge (x-1)>0 \vee (x+4<0) \wedge (x-1<0)$ so we have 4 possibilities.

1) $(x+4>0) \wedge (x-1)>0 \wedge x-2>0 \rightarrow x>-4 \wedge x>1 \wedge x>2 \quad \therefore x>2$
2) $(x+4<0) \wedge (x-1<0) \wedge x-2>0 \rightarrow x<-4 \wedge x<1 \wedge x>2$ Reject
3) $(x+4>0) \wedge (x-1<0) \wedge x-2<0 \rightarrow x>-4 \wedge x<1 \wedge x<2 \quad \therefore -4<x<1$
4) $(x+4<0) \wedge (x-1>0) \wedge x-2<0 \rightarrow x<-4 \wedge x>1 \wedge x<2$ Reject

Solution set $\{x>2 \vee -4<x<1\}$



SUMMARY:

What must be true of two factors to yield a positive product? Negative product?

LESSON: 19

AIM: How do we simplify radical expressions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Simplify radical expressions of index 2 and 3.
2. Add and subtract radicals.

VOCABULARY: unlike radicals, principal square root, principal nth root, radicand, index

MOTIVATION:

For a, b non-negative numbers compare the relative values of expressions 1 and 2.

1) $\sqrt{a+b}$ 2) $\sqrt{a} + \sqrt{b}$

DEVELOPMENT:

1. Ask the students for their responses to the beginning exercise:
2. Have the students choose suitable values of a and b to justify their conjectures.
   For example: i) $\sqrt{9+16} < \sqrt{9} + \sqrt{16}$
   ii) $\sqrt{4+0} = \sqrt{4} + \sqrt{0}$
   iii) $\sqrt{0+25} = \sqrt{0} + \sqrt{25}$
3. Elicit: a) If a, b > 0, then $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$
   b) If a = 0 or b = 0, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
4. Ask the students to evaluate each of the following:
   a) $\sqrt{144}$ b) $\sqrt{.16}$ c) $\sqrt{-64}$ d) $\sqrt[3]{-64}$
5. Define the principal square root of a positive number k, the positive square root, $\sqrt{k}$. Thus $\sqrt{144} = 12$, $\sqrt{.16} = .4$. Additionally have the students note there is no real number equal to $\sqrt{-64}$. However the $\sqrt[3]{-64} = -4$.
6. Ask: What is the principal nth root of a number?
   Define: The principal nth root of a number is the radical, $\sqrt[n]{k}$, where k = the radicand, n = the index, a

natural number that indicates which root is to be taken. In a square root, the index is understood to be 2.

7. Ask: When can we take the principal nth root of a negative radicand?
   Elicit: We can take the principal nth root of a negative radicand only when the index n is odd.
   Examples: $\sqrt[9]{-1} = -1$, $\sqrt[5]{-32} = -2$

8. Ask: If a and b are non-negative compare the values of the radical expressions: $\sqrt{ab}$ and $\sqrt{a} \cdot \sqrt{b}$
   Elicit by illustration that $ab = \sqrt{a} \cdot \sqrt{b}$
   and generalize that (1) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
   (2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

9. Ask: What is the sum of the $\sqrt{48} + \sqrt{3}$
   Elicit: From the beginning exercise we know that it is **not** $\sqrt{48+3} = \sqrt{51}$

10. Ask: When can we combine radicals?
    Elicit: Radicals can only be added when they have the same index and the same radicand.
    Example: (a) $3\sqrt[3]{5} + 6\sqrt[3]{5} - 2\sqrt[3]{5}$
    Using the distributive law: $= (3+6-2)\sqrt[3]{5}$
    $= 7\sqrt[3]{5}$
    (b) $\sqrt[3]{7} - \sqrt{7}$ cannot be combined into a single term for while the radicands are the same the indices are not the same.

11. Have the students recall that radical expressions can be simplified.
    Ask: What is the simplest form of a square-root radical?
    After fielding the students responses inform them:
    The simplest form of a square-root radical is a monomial in the form $k\sqrt{r}$ where k is a non zero rational number and the radicand r contains no perfect square factors greater than 1.

12. Review with the students the technique for simplifying square roots. Have the students recall that by dividing by the non-perfect square factors in ascending order

(2,3,5,6,...) they will find the highest perfect square more quickly.

For example: a) $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$

b) $\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

c) $\sqrt{108} = \sqrt{36 \cdot 3} = \sqrt{36} \cdot \sqrt{3} = 6\sqrt{3}$

13. Ask: For $X \geq 0$, what is the (a) $\sqrt{x^2}$ (b) $\sqrt{x^4}$ (c) $\sqrt{x^6}$ (d) $\sqrt{x^7}$

Have students generalize from these results.

Elicit: 1) For even exponents like 2m: $\sqrt{x^{2m}} = x^m$

2) For odd exponents like 2m+1: $\sqrt{x^{2m+1}} = \sqrt{x^{2m}x}$ $= x^m\sqrt{x}$

14. Have the students find the solution to Application #1.

15. Ask: What would we do differently when we simplify cube roots?

Elicit: Instead of factoring and focusing in on perfect squares we look for perfect cubes.

16. Model with the students the solution to:

$$\sqrt[3]{40x^8y^{12}} = \sqrt[3]{8 \cdot 5 \cdot x^6 \cdot x^2 \cdot y^{12}}$$
$$= \sqrt[3]{8} \cdot \sqrt[3]{5} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{y^{12}}$$
$$= 2x^2y^4\sqrt[3]{5x^2}$$

17. Have the students solve Application #2.

18. Ask: How can simplification aid us in combining unlike radicals? For example $\sqrt{48} + \sqrt{3}$

Elicit: If the indices are the same the radicands can possibly be made alike.

For example: $\sqrt{48} + \sqrt{3}$

$$= \sqrt{16 \cdot 3} + \sqrt{3}$$
$$= 4\sqrt{3} + \sqrt{3}$$
$$= 5\sqrt{3}$$

19. Have the students do Application #3.

20. Ask: How do we simplify radical expressions with fractions?

Model the solution: $\sqrt{\frac{30}{6}} + \frac{\sqrt{20}}{3} - \sqrt{\frac{45}{4}}$

$$= \sqrt{5} + \frac{\sqrt{4\cdot 5}}{3} - \frac{\sqrt{9\cdot 5}}{\sqrt{4}}$$

$$= \sqrt{5} + \frac{2\sqrt{5}}{3} - \frac{3\sqrt{5}}{2}$$

$$= \frac{6\sqrt{5}}{6} + \frac{4\sqrt{5}}{6} - \frac{9\sqrt{5}}{6}$$

$$= \frac{(6+4-9)\sqrt{5}}{6}$$

$$= \frac{\sqrt{5}}{6}$$

21. Have the students do Applications #4, #5 and #6.

APPLICATIONS:

1. Simplify: (a) $2\sqrt{300x^6y^7}$ (b) $6\sqrt{180x^8y^{13}}$

2. Simplify: (a) $\sqrt[3]{-54}$ (b) $8\sqrt[3]{128x^{10}}$

3. Combine: $\sqrt{27} + 2\sqrt{18} + \sqrt{72} - 3\sqrt{8}$

4. Combine: (a) $\sqrt{\frac{24}{3}} + \frac{\sqrt{72}}{3}$

   (b) $-2\sqrt{175} + \sqrt{243} + 5\sqrt{63}$

5. Combine: (a) $\sqrt[3]{108} - \sqrt[3]{32}$

   (b) $7\sqrt{a^9} - 5a^2\sqrt{a^3} - \sqrt{a^7}$

6. Find the perimeter of a rectangle whose length is $\sqrt{75}$ and whose width is $\sqrt{27}$ to the nearest tenth.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $2\sqrt{300x^6y^7}$

$= 2\sqrt{100\cdot 3\cdot x^6y^6\cdot y}$

$= 2\cdot 10\cdot x^3y^3\sqrt{3y}$

$= 20x^3y^3\sqrt{3y}$

b) $6\sqrt{180x^8y^{13}}$

$= 6\sqrt{36\cdot 5\cdot x^8\cdot y^{12}\cdot y}$

$= 6\cdot 6\cdot x^4\cdot y^6\sqrt{5y}$

$= 36x^4y^6\sqrt{5y}$

2. a) $\sqrt[3]{-54}$

$= \sqrt[3]{-27\cdot 2}$

$= -3\sqrt[3]{2}$

b) $8\sqrt[3]{128x^{10}}$

$= 8\sqrt[3]{64\cdot 2x^9\cdot x}$

$= 8\cdot 4\cdot x^3\sqrt[3]{2x}$

$= 32x^3\sqrt[3]{2x}$

3. $\sqrt{27} + 2\sqrt{18} + \sqrt{72} - 3\sqrt{8}$

$= \sqrt{9\cdot 3} + 2\sqrt{9\cdot 2} + \sqrt{36\cdot 2} - 3\sqrt{4\cdot 2}$

$= 3\sqrt{3} + 2\cdot 3\sqrt{2} + 6\sqrt{2} - 3\cdot 2\sqrt{2}$

$= 3\sqrt{3} + 6\sqrt{2} + 6\sqrt{2} - 6\sqrt{2}$

$= 3\sqrt{3} + 6\sqrt{2}$

4. a) $\sqrt{\frac{24}{3}} + \frac{\sqrt{72}}{3}$

$= \sqrt{8} + \frac{\sqrt{36\cdot 2}}{3}$

$= \sqrt{4\cdot 2} + \frac{6\sqrt{2}}{3}$

$= 2\sqrt{2} + 2\sqrt{2}$

$= 4\sqrt{2}$

b) $-2\sqrt{175} + \sqrt{243} + 5\sqrt{63}$

$= -2\sqrt{25\cdot 7} + \sqrt{81\cdot 3} + 5\sqrt{9\cdot 7}$

$= -2\cdot 5\sqrt{7} + 9\sqrt{3} + 5\cdot 3\sqrt{7}$

$= -10\sqrt{7} + 9\sqrt{3} + 15\sqrt{7}$

$= 5\sqrt{7} + 9\sqrt{3}$

5. a) $\sqrt[3]{108} - \sqrt[3]{32}$

$= \sqrt[3]{27\cdot 4} - \sqrt[3]{8\cdot 4}$

$= 3\sqrt[3]{4} - 2\sqrt[3]{4}$

$= \sqrt[3]{4}$

b) $7\sqrt{a^9} - 5a^2\sqrt{a^3} - \sqrt{a^7}$

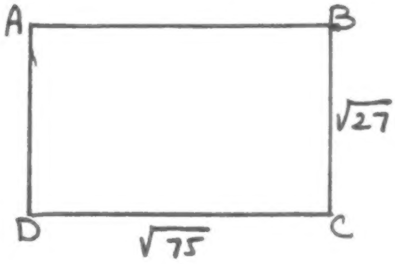
$= 7\sqrt{a^8\cdot a} - 5a^2\sqrt{a^2\cdot a} - \sqrt{a^6\cdot a}$

$= 7a^4\sqrt{a} - 5a^2\cdot a\sqrt{a} - a^3\sqrt{a}$

$= 7a^4\sqrt{a} - 5a^3\sqrt{a} - a^3\sqrt{a}$

$= (7a^4 - 6a^3)\sqrt{a}$

6.



$P = 21 + 2w$

$= 2\sqrt{75} + 2\sqrt{27}$

$= 2\sqrt{25\cdot 3} + 2\sqrt{9\cdot 3}$

$= 2\cdot 5\sqrt{3} + 2\cdot 3\sqrt{3}$

$= 10\sqrt{3} + 6\sqrt{3}$

$= 16\sqrt{3}$

The $\sqrt{3} \approx 1.732$

$P = 16(1.732) = 27.712$

$P \approx 27.7$ units

SUMMARY:

1. When can we combine radical expressions?
2. Make up a radical expression which can be simplified when we take the fifth root.

* 3. If $x < 0$, what is $\sqrt{x^2}$?

LESSON: 20

AIM: How do we multiply and divide radical expressions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Multiply polynomial expressions containing radicals and express them in simplest form.
2. Divide polynomials by monomial expressions containing radicals and express the quotient in simplest form.
3. Express fractions with irrational denominators of the type $\sqrt{a}$ or $a + \sqrt{b}$ with rational denominators.

VOCABULARY: conjugates, rationalizing the denominator

MOTIVATION:

Compute the value of each of the following:

(a) $\frac{1}{1.414}$ (b) $\frac{1.414}{2}$

DEVELOPMENT:

1. Ask, "How do the values computed in the beginning exercise compare?"
   Elicit: The values are approximately equal. (.707)
2. Ask: Which of the two computations was easier?
   Elicit: The division in (a) is more time consuming.
3. Ask, "Which fraction would be easier to find an approximation for?"
   (a) $\frac{\sqrt{2}}{2}$ $\frac{1}{\sqrt{2}}$
   Elicit: In both cases the $\sqrt{2}$ would have to be approximated. (1.414) However, substituting 1.414 for the $\sqrt{2}$ would be the same exercise as the beginning exercise.
4. Ask, "What is the relationship between the $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$?"
   Elicit: They are equivalent fractions written in different forms.

5. Review with the students rational and irrational numbers. Ask, "Give examples of rational and irrational numbers. Emphasize that $\sqrt{64}$, $\sqrt[3]{-27}$ are rational numbers.

6. Ask: Which form of these fractions would make computations simpler?
   Elicit: By having an equivalent fraction with a rational denominator often simplifies computations.

7. Ask: How do we find an equivalent fraction with a rational denominator? For example: Consider $\frac{6}{\sqrt{3}}$
   Elicit: (1) We must find a number which we can multiply the denominator by that will make the product rational. For example: $\sqrt{3} \cdot \sqrt{3} = 3$. (2) In order to keep the value of the fraction the same we multiply the numerator by the same number.

$$\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

8. Model the solution to the following exercise:
   Rationalize the denominator:

$$\sqrt{\frac{25}{12}} = \frac{\sqrt{25}}{\sqrt{12}} = \frac{5}{\sqrt{4 \cdot 3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{2 \cdot 3} = \frac{5\sqrt{3}}{6}$$

or

$$\sqrt{\frac{25}{12}} = \frac{\sqrt{25}}{\sqrt{12}} = \frac{\sqrt{25}}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{36}} = \frac{5\sqrt{3}}{6}$$

9. Have the students do Application #1.

10. Ask, "How do we simplify the product of two radical expressions?
    For example: $5x\sqrt{30xy^2} \cdot 2y\sqrt{6xy}$
    Elicit: We use the commutative property of multiplication and the rule for the product of two radicals.

$$a\sqrt[n]{b} \cdot c\sqrt[n]{d} = ac\sqrt[n]{b} \cdot \sqrt[n]{d} = ac\sqrt[n]{bd}$$

11. Model the solution:

$$\begin{aligned} 5x\sqrt{30xy^2} \cdot 2y\sqrt{6xy} &= 5x \cdot 2y\sqrt{30xy^2 \cdot 6xy} \\ &= 10xy\sqrt{180x^2y^3} \\ &= 10xy\sqrt{36 \cdot 5 \cdot x^2 \cdot y^2 \cdot y} \\ &= 10xy \cdot 6xy\sqrt{5y} \\ &= 60x^2y^2\sqrt{5y} \end{aligned}$$

12. Have the students do Application #2.

13. Ask, "How do we multiply radical expressions containing more then one term?"
Elicit: We use the distributive property.

14. Have the students do Application #3.

15. Ask, "What is different about the last problem in the application above?"
Elicit: The answer is a rational number.

16. Ask, "When will the product of two binomials containing radicals be rational?"
Elicit: "When the binomials are in the following forms:

$$a\sqrt{b} + c\sqrt{d},\ a\sqrt{b} - c\sqrt{d}$$

These expressions are called conjugates of each other."

17. Ask, "What is the conjugate of (a) $5 - 2\sqrt{7}$ (b) $3\sqrt{2} + 5\sqrt{7}$?
Elicit: (a) $5 + 2\sqrt{7}$ (b) $3\sqrt{2} - 5\sqrt{7}$

18. Ask: How can we rationalize the denominator of $\frac{10}{2+\sqrt{3}}$?
Elicit: If we multiply both the numerator and the denominator of the fraction by the conjugate of the denominator, the denominator will be rational.

19. Model the solution:

$$\frac{10}{2+\sqrt{3}} = \frac{10}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{10(2-\sqrt{3})}{(2)^2-(\sqrt{3})^2} = \frac{20-10\sqrt{3}}{4-3} = \frac{20-10\sqrt{3}}{1}$$

20. Have the students do Application #4.

21. Ask, "How do we divide expressions such as $\frac{2\sqrt{7}+8\sqrt{21}}{2\sqrt{7}}$?"

Elicit: We can use the distributive law and the rule for division of radicals.

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}} = \frac{a}{c}\sqrt[n]{\frac{b}{d}}$$

22. Model the solution:

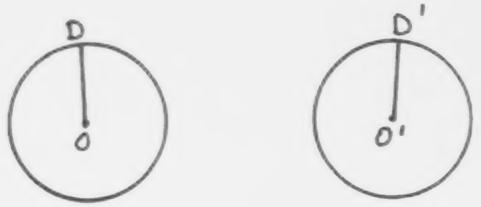
$$\frac{2\sqrt{7}+8\sqrt{2}}{2\sqrt{7}} = \frac{2\sqrt{7}}{2\sqrt{7}} + \frac{8\sqrt{21}}{2\sqrt{7}} = 1\sqrt{\frac{7}{7}} + 4\sqrt{\frac{21}{7}}$$

$$= 1\sqrt{1} + 4\sqrt{3}$$

$$= 1 + 4\sqrt{3}$$

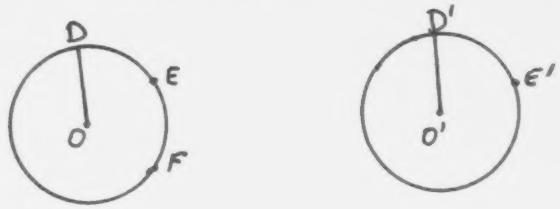
Do Application #5.

10. Definition: Congruent circles are circles with congruent radii.



If $\overline{OD} \cong \overline{O'D'}$, then circle O $\cong$ circle O'

Definition: Congruent arcs are arcs of the same or congruent circles that are equal in measure.



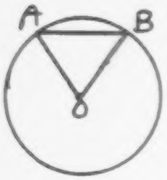
If m $\widehat{DE}$ = m $\widehat{EF}$, then $\widehat{DE} = \widehat{EF}$

If $\overline{OD} \cong \overline{O'D'}$ and m $\widehat{DE}$ = m $\widehat{D'E'}$, then $\widehat{DE} = \widehat{D'E'}$

APPLICATIONS:

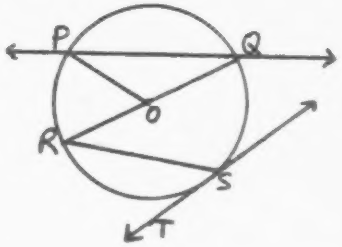
1. Find the measure of a central angle that intercepts an arc whose measure is:
   a) 72 b) 143 c) 45 d) x
2. Find the measure of an arc intercepted by a central angle whose measure is:
   a) 165 b) 120 c) 60 d) y

3. In circle O, $\overline{OA}$ and $\overline{AB}$ are congruent. Find m $\widehat{AB}$.

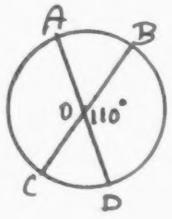


4. With respect to circle O, state the term that best describes each set of points.

$\overline{OR}$
$\overleftrightarrow{PQ}$
∡QOP
$\widehat{PQ}$
$\overline{RS}$
$\overleftrightarrow{ST}$
$\widehat{QRS}$
$\widehat{RPQ}$
$\overline{PQ}$
$\overline{RQ}$
$\widehat{QS}$
∡POR
$\overline{OP}$
$\overline{OQ}$

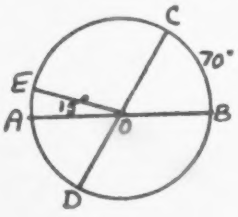


5. Find the measure of each minor arc of the circle.



6. For circle O, find each of the following.

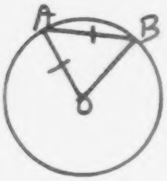
m $\widehat{AE}$
m $\widehat{EC}$
m $\widehat{BC}$
m $\widehat{AD}$
m $\widehat{DB}$
m ∡EOC
m ∡AOD
m ∡DOB
m $\widehat{CBD}$
m $\widehat{CDE}$



SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 72° b) 143° c) 45° d) x°
2. a) 165° b) 120° c) 60° d) y°

3.



$\overline{OA} \cong \overline{AB}$

$\overline{OA} \cong \overline{OB}$ In the same circle, all radii are =

$\therefore \overline{OA} = \overline{AB} = \overline{OB}$

$\triangle$ AOB is equilateral

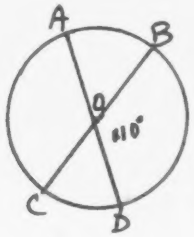
$\measuredangle A \cong \measuredangle AOB \cong \measuredangle B$ $3x = 180$

$\measuredangle AOB = 60°$ $x = 60$

$m\,\widehat{AB} = 60$

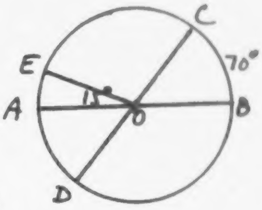
4. $\overline{OR}$ radius $\overline{RS}$ chord $\overline{PQ}$ chord $\overline{OP}$ radius
$\overleftrightarrow{PQ}$ secant $\overleftrightarrow{ST}$ tangent $\overline{RQ}$ diameter $\overline{OQ}$ radius
$\measuredangle QOP$ central angle $\widehat{QRS}$ major arc $\widehat{QS}$ minor arc
$\widehat{PQ}$ minor arc $\widehat{RPQ}$ semicircle $\measuredangle POR$ central angle

5.



$m\,\widehat{BD} = 110$
$m\,\widehat{AC} = 110$
$m\,\widehat{AB} = 180 - 110 = 70$
$m\,\widehat{CD} = 70$

6. $m\,\widehat{AE} = 15$ $m\,\measuredangle EOC = 95$
$m\,\widehat{EC} = 95$ $m\,\measuredangle AOD = 70$
$m\,\widehat{BC} = 70$ $m\,\measuredangle DOB = 110$
$m\,\widehat{AD} = 70$ $m\,\widehat{CBD} = 180$
$m\,\widehat{DB} = 110$ $m\,\widehat{CDE} = 360 - 95 = 265$



SUMMARY:

1. Can a chord be shorter than a radius?...longer than a radius?...longer than a diameter?
2. What is the relationship between a diameter and a radius?
3. What is the relationship between a central angle and its intercepted arc?

LESSONS: 42, 43

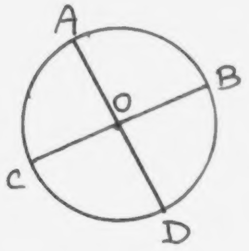
AIM: How do we prove arcs and chords congruent?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State and apply the theorems (prove informally).
   - a) In the same or ≅ circles, ≅ central ∡'s intercept ≅ arcs.
   - b) In the same or ≅ circles, ≅ arcs have ≅ central ∡s.
   - c) In the same circle or in ≅ circles, ≅ central angles have = chords.
   - d) In the same circle or in ≅ circles, ≅ chords have ≅ central angles.
   - e) In the same circle or in ≅ circles, ≅ chords have ≅ arcs.
   - f) In the same circle or in ≅ circles, ≅ arcs have ≅ chords.
2. Apply theorems to algebraic and numeric problems.

VOCABULARY: Review: central ∡, arc, chord, vertical angles.

MOTIVATION:



Given: Circle O with ∡AOB, ∡COD and $\overline{AD}$ and $\overline{CB}$ intersect at O.
What conclusion can we draw about $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$?

Elicit:
- a) ∡AOB and ∡COD are central angles.
- b) ∡AOB and ∡COD are vertical angles.
- c) ∡AOB ≅ ∡COD. Vertical angles are ≅.

DEVELOPMENT:

d) $m \angle AOB = m \overset{\frown}{AB}$ A Central angle is equal to
$m \angle COD = m \overset{\frown}{CD}$ the measure of its intercepted arc.

e) Therefore $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

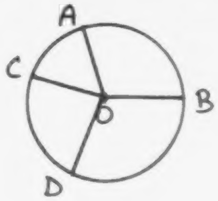
1. Suppose that the two central angles are NOT vertical angles, what conclusion can be drawn?

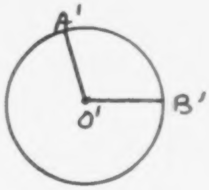
   State and prove informally:

   Theorem: In the same circle or in congruent circles, congruent central angles intercept congruent arcs.

   Given: Circle O $\cong$ Circle O'
   $\angle AOB \cong \angle COD$
   $\angle AOB \cong \angle A'O'B'$

   To Prove: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$
   $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$





** Formal proof is OPTIONAL.

Statements	Reasons
1. Circle O $\cong$ Circle O'	1. Given
2. $\angle AOB \cong \angle COD$ $\angle AOB \cong \angle A'O'B'$	2. Given
3. $m \angle AOB = m \overset{\frown}{AB}$ $m \angle COD = m \overset{\frown}{CD}$ $m \angle A'O'B' = m \overset{\frown}{A'B'}$	3. A central angle is measured by its intercepted arc.
4. $m \overset{\frown}{AB} = m \overset{\frown}{CD}$ $m \overset{\frown}{AB} = m \overset{\frown}{A'B'}$	4. Transitive Property
5. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$ $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$	5. Congruent arcs are arcs of the same or congruent circles that are equal in measure.

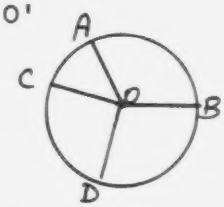
Do application 1A.
Do application 1C.

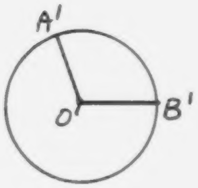
2. State and have students prove informally: ** Formal proof is OPTIONAL

Theorem 2: In a circle or in congruent circle, congruent arcs are intercepted by congruent central angles.

Given: Circle O $\cong$ Circle O'
$\overset{\frown}{AB} = \overset{\frown}{CD}$
$\overset{\frown}{AB} = \overset{\frown}{A'B'}$

Prove: $\measuredangle AOB \cong \measuredangle COD$
$\measuredangle AOB \cong \measuredangle A'O'B'$



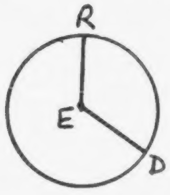


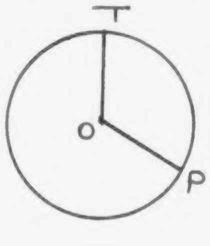
** Formal proof is OPTIONAL.

Statements	Reasons
1. Circle O $\cong$ Circle O'	1. Given
2. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$, $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$	2. Given
3. m $\overset{\frown}{AB}$ = m $\overset{\frown}{CD}$ m $\overset{\frown}{AB}$ = m $\overset{\frown}{A'B'}$	3. Congruent arcs are arcs of the same or congruent circles that are = in measure.
4. m $\overset{\frown}{AB}$ = m $\measuredangle AOB$ m $\overset{\frown}{CD}$ = m $\measuredangle COD$ m $\overset{\frown}{A'B'}$ = m $\measuredangle A'O'B'$	4. The degree measure of an arc is = to the measure of the central angle that intercepts the arc.
5. m $\measuredangle AOB$ = m $\measuredangle COD$ m $\measuredangle AOB$ = m $\measuredangle A'O'B'$	5. Transitive
6. $\measuredangle AOB \cong \measuredangle COD$ $\measuredangle AOB \cong \measuredangle A'O'B'$	6. Congruent angles are angles that have the same measure.

Do Application 1B.

3. * Pose the question:
Are arcs that have equal measures congruent?





Elicit:

$m\angle RED = m\angle TOP$

$\angle RED \cong \angle TOP$

$m\overset{\frown}{RD} = m\overset{\frown}{TP}$

In circles that are not congruent, arcs that have equal measures are not congruent. Thus $\overset{\frown}{RD} \ncong \overset{\frown}{TP}$.

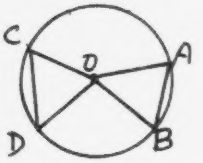
4. Ask the class to draw a conclustion about chords of the same or in $\cong$ circles if they intercept congruent central angles. Elicit:

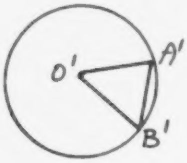
   Theorem: In the same circle or in congruent circles, congruent central angles have congruent chords.

   Given: $\odot O \cong \odot O'$

   $\angle AOB \cong \angle COD$

   $\angle AOB \cong \angle A'O'B'$





   To Prove: $\overline{AB} \cong \overline{CD}$

   $\overline{AB} \cong \overline{A'B'}$

   Hint: Prove triangles OOB, COD, and A'O'B' congruent using SAS $\cong$ SAS. ** Proof is OPTIONAL.

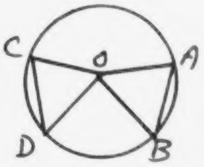
5. Ask the class to draw a conclusion about central angles in the same or in $\cong$ circles if they intercept congruent chords. Elicit:

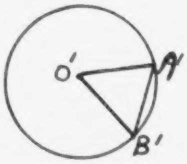
   Theorem: In the same circle or in congruent circles, congruent chords have congruent central angles.

   Given: $\odot O \cong \odot O'$

   $\overline{AB} \cong \overline{CD}$

   $\overline{AB} \cong \overline{A'B'}$





   To Prove: $\angle AOB \cong \angle COD$

   $\angle AOB \cong \angle A'O'B'$

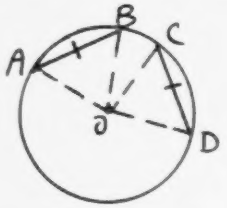
   Hint: Prove triangles AOB, COD, A'O'B' congruent using SSS $\cong$ SSS.

   ** Proof is OPTIONAL.

   Do Application 1D.

6. Ask the class to draw a conclusion about arcs in the same or in congruent circles if they intercept congruent chords. Elicit:

Theorem: In the same circle (or in congruent circles), congruent chords have congruent arcs.



Given: ⊙O, $\overline{AB} \cong \overline{CD}$
Prove: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$
** Proof is OPTIONAL.

Statements	Reasons
1. $\overline{AB} = \overline{CD}$	1. Given
2. Draw radii $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$	2. Two points determine a line.
3. $\overline{OA} \cong \overline{OC}$, $\overline{OB} \cong \overline{OD}$	3. In the same ⊙, all radii are ≅.
4. △ AOB ≅ △ COD	4. SSS ≅ SSS
5. ∡AOB ≅ ∡COD	5. Corresponding parts of ≅ triangles are ≅
6. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$	6. In a ⊙, ≅ central angles intercept congruent arcs.

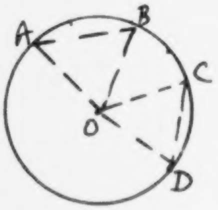
Do Application 1E.

7. Ask the class to draw a conclusion about chords of the same or in ≅ circles if they intercept congruent arcs. Elicit:

Theorem: In the same circle (or in congruent circles) congruent arcs have congruent chords.

Given: ⊙O, $\overset{\frown}{AB} \cong \overset{\frown}{CD}$

Prove: $\overline{AB} \cong \overline{CD}$



** Proof is OPTIONAL.

Hint: Prove $\triangle AOB$ and $\triangle COD$
congruent using SAS $\cong$ SAS.

Do Application 1F.

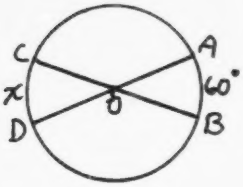
Do Application 1G.

Do Application 1H.

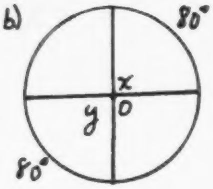
APPLICATIONS:

1. In the following diagrams, O is the center of the circle. Find the values of x, y, z and state the reason.

a)



b)



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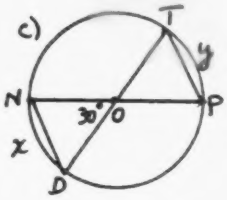

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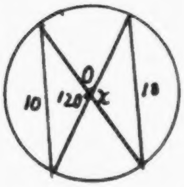
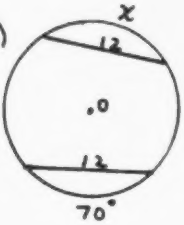
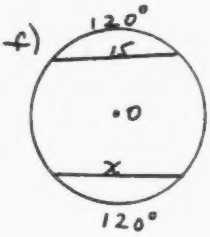


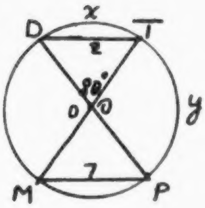
e)



f)


120°
15
O
x
120°

g)


x
D
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80°
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y
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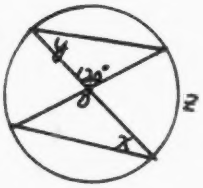
h)



SOLUTIONS TO APPLICATION PROBLEMS:

a) $\measuredangle AOB = 60^\circ$ The central angle is equal to the measure of its intercepted arc.

$\measuredangle COD = 60^\circ$ Vertical angles are congruent.

$\overset{\frown}{CD} = 60$ In the same or ≅ circles, ≅ central angles intercept congruent arcs.

b) $\measuredangle x = 80^\circ$ In the same or ≅ circles, ≅ arcs

$\measuredangle y = 80^\circ$ have ≅ central ∢s.

c) $\measuredangle TOP = 30^\circ$ Vertical angles are congruent.

$\overset{\frown}{ND} \cong \overset{\frown}{TP} = 30^\circ$ In the same or ≅ circles, ≅ central angles intercept congruent arcs.

d) $\measuredangle x = 120^\circ$ In the same circle or in ≅ circles, ≅ chords have ≅ central angles.

e) $x = 70^\circ$ In the same circle or ≅ circles, ≅ chords have ≅ arcs.

f) $x = 15$ In the same circle or in ≅ circles, ≅ arcs have ≅ chords.

g) $\measuredangle MOP = 80^\circ$ Vertical angles are congruent.

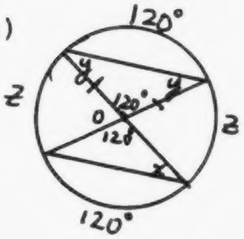
$z = 7$ In the same circle or in ≅ circles, ≅ central angles have ≅ chords.

$x = \overset{\frown}{DT} = 80^\circ$ The central angle is equal to the measure of its intercepted arc.

$\overset{\frown}{MP} = 80^\circ$ In the same or ≅ circles, ≅ central ∢s intercept ≅ arcs.

$$80^\circ+80^\circ+y+y = 360^\circ$$
$$160^\circ+2y = 360^\circ$$
$$2y = 200^\circ$$
$$y = 100^\circ$$

* h)



$$y+y+120^\circ = 180^\circ$$
$$2y+120^\circ = 180^\circ$$
$$2y = 60^\circ$$
$$y = 30^\circ$$
$$x = 30^\circ$$

$$z+z+120^\circ+120^\circ = 360^\circ$$
$$2z+240^\circ = 360^\circ$$
$$2z = 120^\circ$$
$$z = 60^\circ$$

SUMMARY:

Ask the class:

Name the methods of proving arcs in the same or congruent circles congruent.

Elicit:

To prove that arcs in the same or in congruent circles are congruent, prove that either one of the following statements is true:

1) The central angles of the arcs congruent.
2) The chords of the arcs congruent.

LESSONS: 44, 45

AIM: How do we prove chords congruent?

PERFORMANCE OBJECTIVES: The student will be able to...

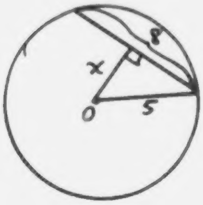
1. State and apply the theorems (prove informally).
   - a) A diameter perpendicular to a chord bisects the chord and both pairs of arcs.
   - b) A line through the center of a circle which bisects a non-diameter chord is perpendicular to it.
   - c) In the same circle or in ≅ circles, ≅ chords are equally distant from the center.
   - d) In the same circle or in ≅ circles, chords equally distant from the center are ≅.
2. Apply indicated theorems to algebraic and numeric problems.

VOCABULARY: Review diameter, chord, bisect, arc

MOTIVATION:

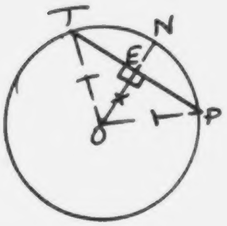
A chord of length 8 is drawn to a circle of radius 5. Find the distance from the chord to the center of the circle.

How do we find the distance from a point to a line?
Answer: The distance from a point to a line is the length of the perpendicular from the point to the line.



What conclusions can be drawn?

Consider the following: ______________________________



$\overline{OE} \perp \overline{TP}$

$\overline{OP} \cong \overline{OT}$ Radii of same circle are congruent.

$\overline{OE} \cong \overline{OE}$ Identity

$\triangle TOE = \triangle POE$ Hy leg ≅ Hy leg

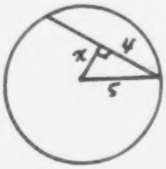
Conclusions:

a) $\overline{TE} \cong \overline{EP}$ Corresponding parts of congruent triangles are congruent.

b) Central angle TOE $\cong$ Central angle POE

c) $\overset{\frown}{TN} = \overset{\frown}{NP}$

Return to motivation



$x^2+4^2 = 5^2$

$x^2+16 = 25$

$x^2 = 9$

$x = \pm 3 \qquad x = 3$

$x = -3$

reject $x=-3$

$\therefore x=3$

DEVELOPMENT:

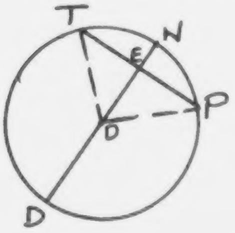
1. Extend $\overline{ON}$ through O to make diameter $\overline{DON}$.

   What conclusion can be drawn?

   Elicit theorem:

   A diameter perpendicular to a chord bisects the chord and both pairs of arcs.

   ** Proof is OPTIONAL



Given: $\odot O$; diameter DN, chord TP

$\overline{DN} \perp \overline{TP}$

Prove: $\overline{TE} \cong \overline{EP}$

$\overset{\frown}{TN} \cong \overset{\frown}{NP}$

$\overset{\frown}{TD} \cong \overset{\frown}{PD}$

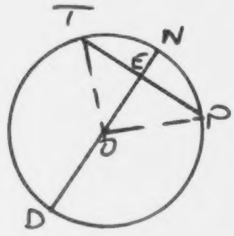
* Honor students can complete the PROOF.

	Statements		Reasons
1.	$\overline{DN} \perp \overline{TP}$	1.	Given
2.	∢TOE and ∢POE are right angles	2.	Perpendicular lines intersect to form right angles
3.	Draw $\overline{OT}$ and $\overline{OP}$	3.	Two points determine a line
4.	$\overline{OT} \cong \overline{OP}$	4.	

5.	$\overline{OE} \cong \overline{OE}$	5.	
6.	$\triangle TOE = \triangle POE$	6.	
7.	$\overline{TE} \cong \overline{EP}$	7.	
8.	$\measuredangle TOE \cong \measuredangle POE$	8.	
9.	$\overset{\frown}{TN} = \overset{\frown}{NP}$	9.	In a ⊙, ≅ central ∡s have ≅ arcs
10.	∡TOD is supplementary to ∡TOE. ∡POD is supplementary to ∡POE	10.	
11.	$\measuredangle TOD \cong \measuredangle POD$	11.	
12.	$\overset{\frown}{TD} \cong \overset{\frown}{PD}$	12.	In a circle, congruent central angles have congruent arcs.

2. Do Applications 1A.
1B.
1C.

3. Consider a diameter which bisects a chord.



Given: Diameter $\overline{DN}$

$\overline{DN}$ bisects $\overline{TP}$.

What conclusions can we make?

Draw $\overline{OT}$ and $\overline{OP}$

$\overline{OT} \cong \overline{OP}$ All radii of same circle are congruent.

$\overline{OE} \cong \overline{OE}$

$\overline{TE} \cong \overline{EP}$

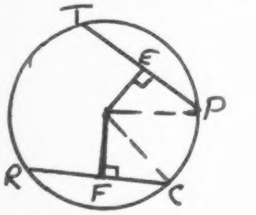
$\therefore\ \triangle TOE = \triangle POE$	SSS $\cong$ SSS
$\angle OET \cong \angle PEO$	Corresponding parts of $\cong$ triangles are $\cong$.
$\angle OET$ is a right angle	
$\angle PEO$ is a right angle	

$\therefore\ \overline{DN} \perp \overline{TP}$

Elicit the theorem: A line through the center of a circle which bisects a nondiameter chord is perpendicular to it.

** Proof is OPTIONAL.

4. Consider two congruent chords in a circle, what is their relationship to the center of the circle.



Draw $\overline{OP}$ and $\overline{OC}$

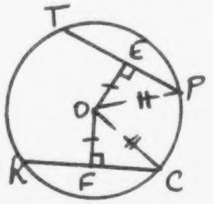
$\overline{OP} \cong \overline{OC}$	All radii of same circle are congruent.
$\overline{OE} \perp \overline{TP}$	
$\overline{OF} \perp \overline{RC}$	
$\overline{PE} \cong \overline{FC}$.	Halves of congruent segments are congruent.
$\triangle POE \cong \triangle FOC$.	Hy leg $\cong$ Hy leg
$\overline{OE} \cong \overline{OF}$	Corresponding parts of congruent triangles are congruent.

Elicit the theorem: In the same circle or in $\cong$ circles, congruent chords are equally distant from the center.

** Formal Proof is OPTIONAL

Do Application 2A.

5. Consider the converse of the above theorem. In the same ⊙ or in = ⊙s, if two chords are equidistant from the center, what is their relationship?



Draw $\overline{OP}$ and $\overline{OC}$

$\overline{OP} \cong \overline{OC}$. All radii of same circle are congruent.

$\overline{OE} \cong \overline{OF}$ Given $\overline{OE} \perp \overline{TP}$; $\overline{OF} \perp \overline{RC}$

$\triangle POE \cong \triangle FOC$. Hy leg ≅ Hy leg

$\overline{PE} \cong \overline{FC}$ Corresponding parts of = s are =.

$PE = \frac{1}{2}TP$ A diameter perpendicular to a chord bisects the chord

$FC = \frac{1}{2}RC$ A diameter perpendicular to a chord bisects the chord.

$\overline{TP} \cong \overline{RC}$ Doubles of congruents are congruent.

Elicit the theorem: In the same circle or in ≅ circles, chords equidistant from the center are congruent.

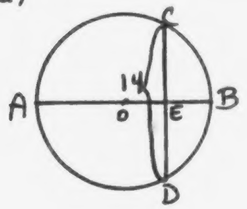
** Proof is OPTIONAL.

Do Application 2B.

APPLICATIONS:

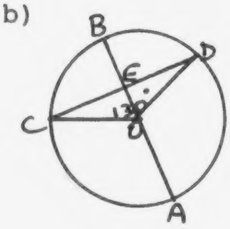
1. Diameter $\overline{AB} \perp \overline{CD}$.

a)



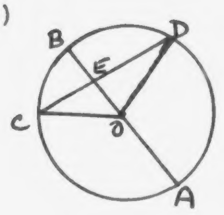
Find length of CE

b)



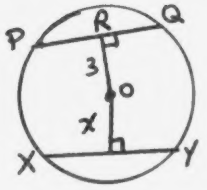
Find m $\overset{\frown}{CD}$, m $\overset{\frown}{BC}$, m $\overset{\frown}{DB}$, m $\overset{\frown}{CAD}$, m $\overset{\frown}{CA}$

c)

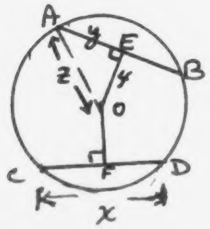


CD = 6, OE = 4
Find length of OD

2.



a) $\overline{PQ} \cong \overline{XY}$
Find the value of x



b) AB = 10, $\overline{AB} \cong \overline{CD}$
Find the values of x, y, z

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) CE = 7
   b) m $\overset{\frown}{CD}$ = 130
      m $\overset{\frown}{BC}$ = 65
      m $\overset{\frown}{DB}$ = 65
      m $\overset{\frown}{CAD}$ = 360-130 = 230
      m $\overset{\frown}{CA}$ = 180-65 = 115
   c) OD = 5

2. a) $x = 3$ In the same circle or in $\cong$ circles, $\cong$ chords are equally distant from the center.

   b) $y = 5$

   $x = 10$

   $4^2+5^2 = z^2$

   $16+25 = z^2$

   $41 = z^2$

   $\pm\sqrt{41} = z$ $z = +\sqrt{41}$

   $z = -\sqrt{41}$ reject

SUMMARY:

1. When are chords equidistant from the center of a circle?
2. Name methods of proving chords in the same or congruent circles congruent.

   Elicit:

   To prove that two chords in a circle or in congruent circles are congruent, prove that either one of the following statements is true.

   a) The arcs of the chords are congruent.

   b) The chords are equidistant from the center of the circle.

LESSONS: 46, 47

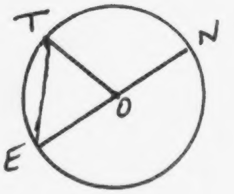
AIM: How do we measure an inscribed angle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define inscribed angle.
2. State and apply the theorems:
   (Prove informally)
   - a) An inscribed angle is measured by one-half the measure of its intercepted arc. Understand all three cases.
   - b) In the same or ≅ circles, if two inscribed ∡s intercept the same arc or ≅ arcs, they are ≅.
   - c) An ∡ inscribed in a semi-circle is a right ∡.
   - d) If a quadrilateral is inscribed in a circle, opposite ∡s are supplementary.
   - e) In a circle, parallel chords intercept ≅ arcs between them.
3. Solve algebraic and numeric exercises with inscribed angles.

VOCABULARY: inscribed angle, parallel chords, inscribed in a circle

MOTIVATION:



Identify a central angle.
Is ∡ TEN a central angle?
Where is its vertex?

Ask class to describe ∡TEN?

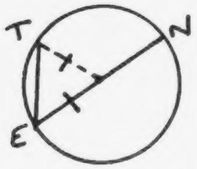
Elicit the definition of an inscribed angle.

An inscribed angle is an angle whose vertex lies on a circle and whose sides contain chords of a circle.

DEVELOPMENT:

1. How do we measure an inscribed angle?

Consider motivation example: The center of the circle was on the inscribed angle?



Thus in isosceles $\triangle$ TOE, m $\measuredangle$ETO $\cong$ m $\measuredangle$TEO. Why?

If two sides of a $\triangle$ are congruent, the angles opposite these sides are congruent.

Let m $\measuredangle$ETO $\cong$ m $\measuredangle$TEO = x.

Then m $\measuredangle$TON = 2x Why?

An exterior angle of a triangle is = to the sum of the two nonadjacent interior angles.

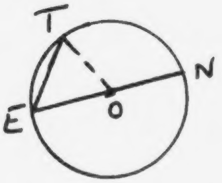
m $\measuredangle$TON = m $\overset{\frown}{TN}$ = 2x. Why? A central angle is measured by its intercepted arc. Thus m $\measuredangle$TEN = $\frac{1}{2}$(2x) = $\frac{1}{2}$ m $\overset{\frown}{TN}$.

What seems to be the relationship between an inscribed angle and its intercepted arc?

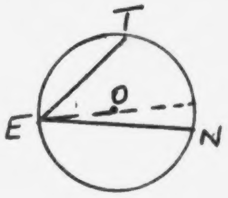
Elicit: An inscribed angle is measured by one half the measure of its intercepted arc.

2. There are three cases to consider in the PROOF depending on whether the center of the circle is on, inside, or outside the inscribed angle.

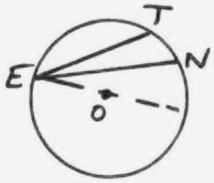
The three cases are illustrated in the diagrams below.



Case I
The center of the circle is contained in one ray of the inscribed angle.



Case II
The center of the circle is inside the inscribed angle.



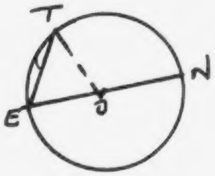
Case III
The center of the circle is outside the inscribed angle.

** The proofs for Case I, Case II, and Case III are optional.

Do Applications 1a to e.
2a to e.
3a to d.

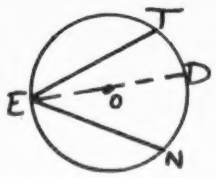
3. Case I. The center of the circle is contained in one ray of the inscribed angle.

Given: $\angle TEN$ is inscribed in $\odot O$
O lies on $\overline{EN}$
Prove: $m\angle TEN = \frac{1}{2} m\,\overset{\frown}{TN}$



Statements	Reasons
1. Draw $\overline{OT}$	1. Two points determine a line.
2. $\overline{OT} \cong \overline{OE}$	2. In a circle, all radii are $\cong$.
3. $\angle TEN \cong \angle OTE$ $m\angle TEN = m\angle OTE$	3. If two sides of a $\triangle$ are $\cong$, then the angles opposite these sides are $\cong$.
4. $m\angle TEN + m\angle OTE = m\angle TON$	4. The measure of an exterior $\angle$ of a $\triangle$ is = to the sum of the measure of the 2 nonadjacent interior $\angle$s of the $\triangle$.
5. $m\angle TEN + m\angle TEN = m\angle TON$ or $2m\angle TEN = m\angle TON$	5. Substitution
6. $m\angle TEN = \frac{1}{2} m\angle TON$	6. Division Property of Equality.
7. $m\angle TON = m\overset{\frown}{TN}$	7. The measure of a central $\angle$ of a $\odot$ is = to the measure of its intercepted arc.
8. $m\angle TEN = \frac{1}{2} m\overset{\frown}{TN}$	8. Substitution

4. Case II. The center of the circle is a point in the interior of the inscribed angle.

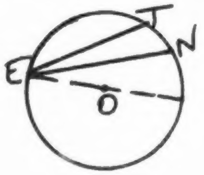


Given: $\angle NET$ is inscribed in $\odot O$, with O in the interior of $\angle NET$

Prove: $m\angle NET = \frac{1}{2}m\overset{\frown}{NT}$

Statements	Reasons
1. $\angle NET$ is inscribed in in circle O; O is a point in the interior of < NET	1. Given
2. Draw $\overline{EO}$ intersecting the circle at D.	2. Two points determine a line.
3. $m\angle NET = m\angle NED + m\angle DET$	3. The whole is equal to the sum of its parts.
4. $m\angle NED = \frac{1}{2}m\overset{\frown}{ND}$ $m\angle DET = \frac{1}{2}m\overset{\frown}{DT}$	4. Case I.
5. $m\angle NET = \frac{1}{2}m\overset{\frown}{ND} + \frac{1}{2}m\overset{\frown}{DT}$	5. Substitution
6. $m\angle NET = \frac{1}{2}m(\overset{\frown}{ND} + m\overset{\frown}{DT})$	6. Distributive Property
7. $m\angle NET = \frac{1}{2}m\overset{\frown}{NT}$	7. Addition Postulate

5. Case III. The center of the circle is a point in the exterior of the inscribed angle.



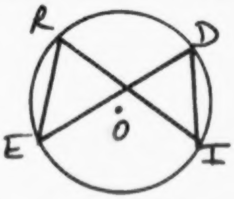
Given: $\angle TEN$ inscribed in circle O, with O in the exterior of $\angle$ TEN.

Prove: $m\angle TEN = \frac{1}{2}m\overset{\frown}{TN}$

Statements	Reasons
1. $\angle TEN$ inscribed in circle O; O is a point in the exterior of $\angle TEN$.	1. Given
2. Draw $\overleftrightarrow{OE}$ intersecting the circle at D.	2. Two points determine a line.
3. $m\angle TEN + m\angle NED = m\angle TED$	3. The whole is = to the sum of its parts.
4. $m\angle TEN = m\angle TED - m\angle NED$	4. Addition property of equality.

5.	$m\angle TED = \frac{1}{2}m\,\overset{\frown}{TD}$ $m\angle NED = \frac{1}{2}m\,\overset{\frown}{ND}$	5.	Case I.
6.	$m\angle TEN = \frac{1}{2}m\overset{\frown}{TD}-\frac{1}{2}m\overset{\frown}{ND}$	6.	Substitution
7.	$m\angle TEN = \frac{1}{2}(m\overset{\frown}{TD}-m\overset{\frown}{ND})$	7.	Distributive Property
8.	$m\angle TEN = \frac{1}{2}\overset{\frown}{TN}$	8.	Substitution

6. Consider two inscribed angles that intercept the same arc. What conclusions can be drawn?



Inscribed angle RED = $\frac{1}{2}\overset{\frown}{RD}$

Inscribed angle RID = $\frac{1}{2}\overset{\frown}{RD}$

Therefore $m\angle RED = m\angle RID$

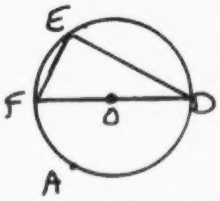
and $\angle RED \cong \angle RID$

What have we shown?

Elicit: Is the same or = circles, if two inscribed $\angle$ s intercept the same arc or $\cong$ arcs, they are $\cong$.

Do Application 4a.

7. Consider inscribed angle FED where FD is a diameter. What conclusions can be drawn?



Remember: A diameter divides a circle into 2 = parts

Thus $\overset{\frown}{FAD}$ is a semicircle.

$m\,\overset{\frown}{FAD} = 180$

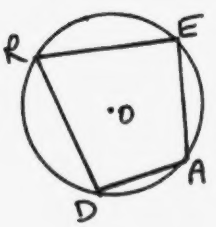
$m\angle FED = \frac{1}{2}\,m\,\overset{\frown}{FAD} = \frac{1}{2}(180) = 90$

What have we shown?

Elicit: An angle inscribed in a semicircle is a right angle.

Do Application 4b.

8. Consider a quadrilateral inscribed in a circle.



Explain: If a quadrilateral is inscribed in a circle, the vertices of the quadrilateral are points of the circle and the side of the quadrilateral are chords of the circle.

What conclusions may be drawn?

Inscribed $\measuredangle DRE = \frac{1}{2}\overset{\frown}{DAE}$

Inscribed $\measuredangle DAE = \frac{1}{2}\overset{\frown}{DRE}$

$\overset{\frown}{DAE} + \overset{\frown}{DRE} = 360°$

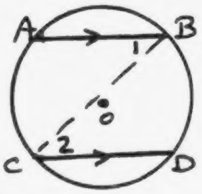
$$\begin{aligned}\measuredangle DRE + \measuredangle DAE &= \tfrac{1}{2}\overset{\frown}{DAE} + \tfrac{1}{2}\overset{\frown}{DRE}\\ &= \tfrac{1}{2}(\overset{\frown}{DAE} + \overset{\frown}{DRE})\\ &= \tfrac{1}{2}(360)\\ &= 180°\end{aligned}$$

Similarly $\measuredangle REA + \measuredangle RDA = 180°$ What have we shown?

Elicit: If a quadrilateral is inscribed in a circle, opposite angles are supplementary.

Do Application 4c.

9. Consider the following diagram with chord AB // chord CD.



What is the relationship between $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$?

Draw BC and note that $\measuredangle 1$ and $\measuredangle 2$ are alternate interior angles.

$\measuredangle 1 \cong \measuredangle 2$	Alternate interior angles of parallel lines are congruent.
$\measuredangle 1 = \frac{1}{2}m\overset{\frown}{AC}$	An inscribed angle is measured by $\frac{1}{2}$ its intercepted arc.

$\measuredangle 2 = \frac{1}{2} m\overset{\frown}{BD}$ An inscribed angle is measured by $\frac{1}{2}$ its intercepted arc.

$\therefore \overset{\frown}{AC} \cong \overset{\frown}{BD}$ Substitution

What have we shown?

Elicit: In a circle, parallel chords intercept congruent arcs between them.

Do Application 4d.

Do Application 5.

APPLICATIONS:

1. Find the measure of an inscribed angle that intercepts an arc whose measure is:

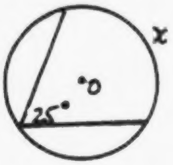
   a) 72° b) 90° c) 180° d) 120° e) x°

2. Find the measure of the arc intercepted by an inscribed angle whose measure is:

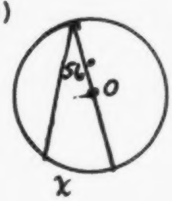
   a) 45° b) 60° c) 86° d) x° e) 2x°

3. Find the value of x.

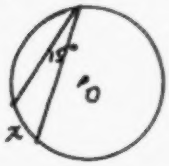
a)



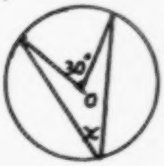
b)



c)

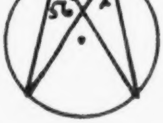


d)

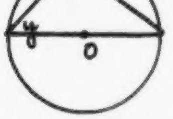


4. Find the value(s) of x, y. State the reason for your answer.

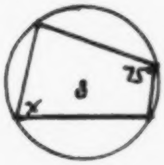
a)



b)



c)



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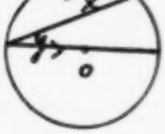
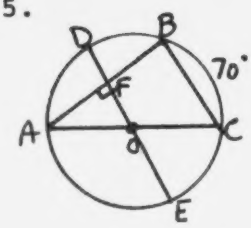


5.



Diameter $\overline{DOE} \perp$ chord $\overline{AB}$ at F, $\overline{AOC}$ is a diameter, and $\overline{BC}$ is a chord of circle O.

m $\overset{\frown}{BC}$ = 70

Find m $\overset{\frown}{AB}$, m ∡AOD,
m ∡A, m $\overset{\frown}{CE}$
m ∡C
m AD

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 36° b) 45° c) 90° d) 60° e) $\frac{x}{2}$°

2. a) 90° b) 120° c) 172° d) 2x° e) 4x°

3. a) x = 50°
   b) x = 112°
   c) x = 30°
   d) x = 15°

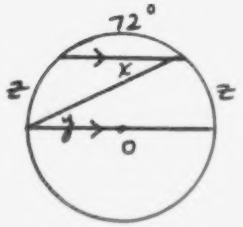
4. a) x = 56° — In the same or = circle, if two inscribed ∡s intercept the same arc or ≅ arcs, they are ≅.

   b) ∡y = 40° — An inscribed angle is measured by one half the measure of its intercepted arc.

   ∡x = 90° — An ∡ inscribed in a semi-circle is a right ∡.

   c) ∡x = 180°-75° = 105° — If a quadrilateral is inscribed in a circle, opposite ∡s are supplementary.

   d)



In a circle, parallel chords intercept ≅ arcs between them.

$$z + 72 + z = 180$$
$$2z + 72 = 180$$
$$2z = 108$$
$$z = 54$$

$\angle x^\circ = 27$, $\angle y^\circ = 27$

An inscribed angle is measured by one half the measure of its intercepted arc.

5. $m\ \widehat{AB} = 180-70 = 110$
$m\ \angle AOD = 55$
$m\ \angle A = 35$
$m\ \widehat{CE} = 55$
$m\ \angle C = 55$
$m\ \widehat{AD} = 55$

SUMMARY:

1. What is the relationship between an inscribed angle and a central angle which intercepts the same arc?

* 2. Prove that a parallelogram inscribed in a circle is a rectangle.

LESSONS: 48, 49

AIM: What happens when tangents to a circle are drawn?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define tangent to a circle.
2. State postulate.
   a) At a given point on a circle, there is one and only one tangent to a circle.

   State theorems.

   b) If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

   c) If a line is perpendicular to a radius at its outer endpoint, then it is a tangent.

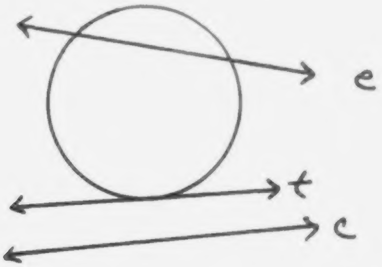
   d) State theorem "2 tangents to a circle from the same external point are $\cong$ (prove informally).
3. Distinguish between tangent circles and common tangents.
4. Apply indicted postulate and theorems to algebraic and numeric problems.

VOCABULARY: tangent to a circle, tangent circles, common tangents

MOTIVATION:

How many points may a line have in common with a circle? Explain.

Draw the possibilities. Label the diagram.



Elicit

A line may have two points, one point or on points in common with a circle.

DEVELOPMENT:

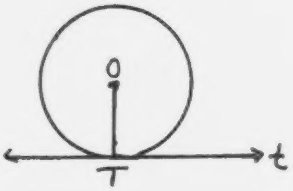
1. In the diagram line t is tangent to circle O because the line intersects the circle at only one point T.

   Have students recall the definition of TANGENT.

   A tangent to a circle is a line which intersects a circle at one and only one point. Explain to the class that the point at which the tangent intersects the circle is called the "point of tangency" or "point of contact". In the diagram, line t is a tangent to circle O. Point T is the point of tangency or point of contact.

2. How many tangents may be drawn at a given point on a circle? Elicit and state the POSTULATE:

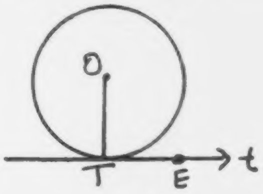
   At a given point on a circle, there is one and only one tangent to the circle.



3. What seems to be the relationship between the tangent and the radius drawn to the point of contact? Elicit and state the theorem:

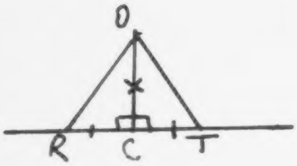
   If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

   ** PROOF IS OPTIONAL



Given: Circle O, tangent $\overleftrightarrow{TE}$, radius $\overline{OT}$

Prove: $\overline{OT} \perp \overleftrightarrow{TE}$



Indirect Proof

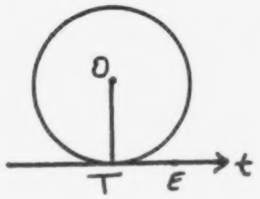
Through point O, there is only one line perpendicular to $\overleftrightarrow{TE}$. Assume that $\overline{OT}$ is not $\perp$ to $\overleftrightarrow{TE}$. Then $\overleftrightarrow{TE}$ contains a point C, other than T, such that $\overline{OC} \perp \overleftrightarrow{TE}$. There is also a point R such that $\overline{CR} \cong \overline{TC}$.

$Rt\angle RCO \cong Rt\angle OCT$, $\overline{OC} = \overline{OC}$, $\triangle ROC \cong \triangle TOC$ (SAS $\cong$ SAS). Therefore $\overline{OR} = \overline{OT}$. Thus tangent $\overleftrightarrow{TE}$ intersects circle O at both R and T. Since the assumption $\overline{OT}$ is not $\perp$ to $\overleftrightarrow{TE}$ leads to this contradiction, the assumption is incorrect. Therefore $\overline{OT} \perp \overleftrightarrow{TE}$.

4. Do Applications #1, #3, #4.

5. What seems to be true if a line is $\perp$ to a radius at its outer endpoint? Elicit and state the theorem.
   If a line is $\perp$ to a radius at its outer endpoint, then it is a tangent.

   ** PROOF IS OPTIONAL

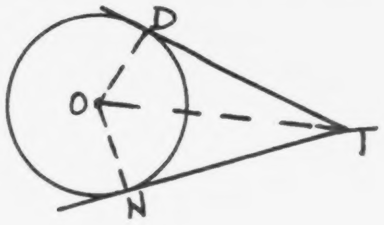


Given: $\overleftrightarrow{TE} \perp$ radius $\overline{OT}$ of circle O
Prove: $\overleftrightarrow{TE}$ is tangent to circle

Indirect Proof:
Given $\overleftrightarrow{TE} \perp$ to radius $\overline{OT}$ of circle O, assume that $\overleftrightarrow{TE}$ also intersects the circle at a second point C.

* HONOR STUDENTS CAN FINISH PROOF.

6. Consider the diagram. $\overleftrightarrow{DT}$ and $\overleftrightarrow{NT}$ are tangents to circle O at D and N respectively?



What conclusions can we make? Explain.

a) $\overleftrightarrow{DT} \perp \overline{OD}$ (If a line is tangent to a circle, it is
   $\overleftrightarrow{NT} \perp \overline{ON}$ perpendicular to the radius drawn to the point of contact.)

b) $\overline{OD} \cong \overline{ON}$ (All radii of same circle are congruent.)

c) $\overline{OT} \cong \overline{OT}$ (Identity)

d) $\angle ODT$ and $\angle ONT$ are right angles. (Perpendicular lines intersect and form right angles.)

e) R+ $\triangle$ DOT = R+ $\triangle$ NOT (Hy leg $\cong$ hy leg.)

f) $\overleftrightarrow{DT} \cong \overleftrightarrow{DN}$ (Corresponding parts of $\cong$ triangles are congruent.)

g) $\angle DTO \cong \angle NTO$ (Corresponding parts of $\cong$ triangles are congruent.)

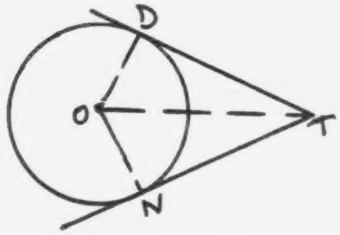
SUMMARY: What have we shown?

Elicit:

I) Two tangents to a circle from the same external point are $\cong$.

II) If two tangents are drawn to a circle from an external point, the line determined by that point and the center of the circle bisects the angle formed by the tangents. Since $\angle DTO$ and $\angle NTO$ are congruent, then $\overline{TO}$ bisects $\angle DTN$.

Theorem: Two tangents to a circle from the same external point are =.



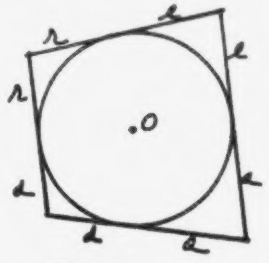
Given: $\overleftrightarrow{DT}$ tangent to circle O at D.
$\overleftrightarrow{NT}$ tangent to circle O at N.
Prove: $\overline{DT} \cong \overline{NT}$

** FORMAL PROOF IS OPTIONAL

Plan: Draw $\overline{OD}$, $\overline{OT}$, $\overline{ON}$. Prove right triangles ODT and ONT congruent by hy leg $\cong$ hy leg.

7. Do Applications #3, #5a, 5b.

* 8. Show that if a quadrilateral is circumscribed about a circle, then the sum of the lengths of each pair of opposite sides is equal.



Label the diagram. Remember: Two tangents to a circle from the same external point are congruent.

The sum of the lengths of each pair of opposite sides is

r + e + a + d

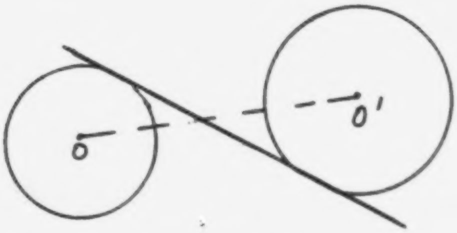
These sums are equal.

r + e + a + d = r + e + a + d

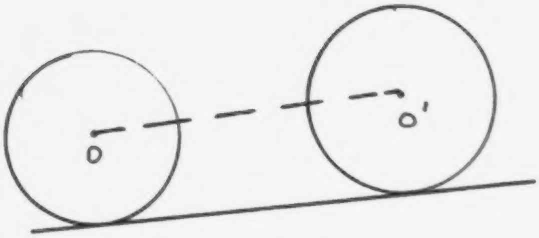
9. Define a common tangent.

A common tangent is a line that is tangent to each of two circles.

If the tangent intersects the line segment joining the two centers, it is a common internal tangent. If it does not, it is a common external tangent.



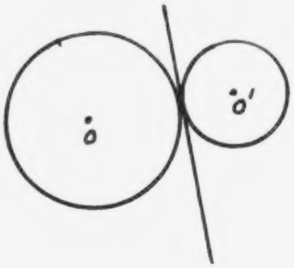
common internal tangent



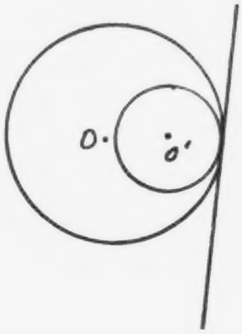
common external tangent

Definition:

Two circles are tangent to each other if they are tangent to the same line at the same point.



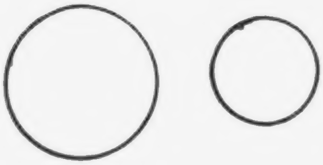
Externally Tangent Circles



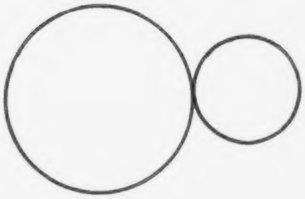
Internally Tangent Circles

10. Sketch these diagrams and state the number of common tangents which can be drawn.

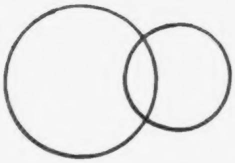
a)



b)



c)



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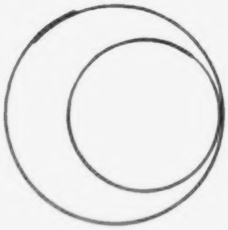
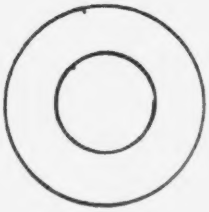


e)

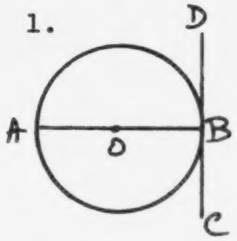


Answers a) four
b) three
c) two
d) one
e) no

11. Do Application #6.

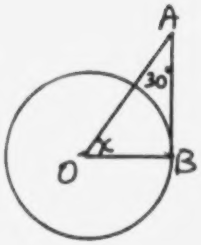
APPLICATIONS:

1.



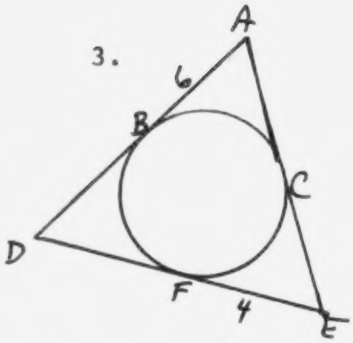
$\overleftrightarrow{DC}$ is tangent
to ⊙O at B
Find $m\angle ABC$

2.



$\overleftrightarrow{AB}$ is tangent
to ⊙O at B
Find value of x

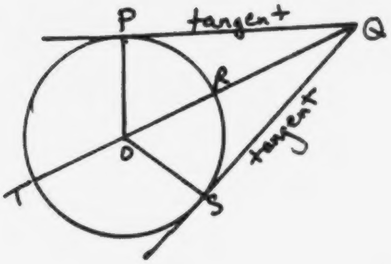
3.



$\overleftrightarrow{If}$ AB = 6, EF = 4,
find AE

4. Given circle O with $\overleftrightarrow{QR}$ tangent to the circle at R. If $m\angle OQR = 20$, find $m\angle ROQ$.

5.



a) If OP = 15 and PQ = 20, find $\overline{OQ}$ and $\overline{OS}$.

b) If PQ = 4x-1, SQ = x+11, OP = 2x, find x, $\overline{PQ}$, $\overline{OP}$ and $\overline{OQ}$.

Given:

$\overleftrightarrow{PQ}$ is tangent to $\odot O$ at P.
$\overleftrightarrow{QS}$ is tangent to $\odot O$ at S.

6. Draw a sketch of two circles having two common external tangents and one common internal tangent.

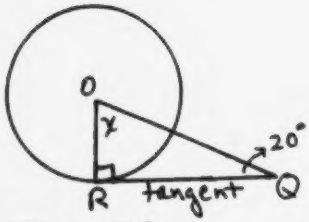
SOLUTIONS TO APPLICATION PROBLEMS:

1. m ∡ABC = 90°

2. $\measuredangle x + 90 + 30 = 180$
$\measuredangle x + 120 = 180$
$\measuredangle x = 60$

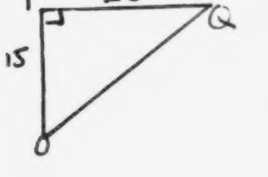
3. If AB = 6, then AC = 6
If EF = 4, then EC = 4
$\overline{AE} = \overline{AC} + \overline{EC}$
$6 + 4 = 10$

4.



$\measuredangle ORQ = 90°$
$\measuredangle x + 90 + 20 = 180$
$\measuredangle x + 110 = 180$
$\measuredangle x = 70$
$\measuredangle ROQ = 70$

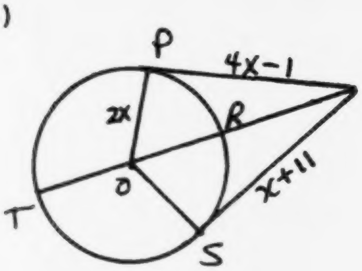
5. a)



Pythagorean Theorem
$15^2 + 20^2 = (OQ)^2$
$225 + 400 = (OQ)^2$
$625 = (OQ)^2$
$25 = OQ$

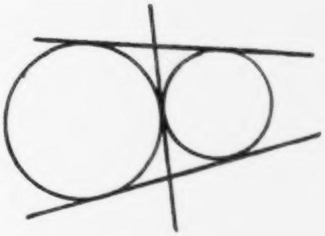
$\overline{OP} \cong \overline{OS}$ All radii of same circle are congruent
OS = 15

b)



$$4x - 1 = x + 11$$
$$3x = 12$$
$$x = 4$$
$$PQ = 4x-1 = 4(4)-1 = 16-1 = 15$$
$$OP = 2x = 2(4) = 8$$
$$(\overline{OQ})^2 = (\overline{PQ})^2 + (\overline{OP})^2$$
$$= (15)^2 + (8)^2$$
$$= 225 + 64$$
$$= 289$$
$$OQ = 17$$

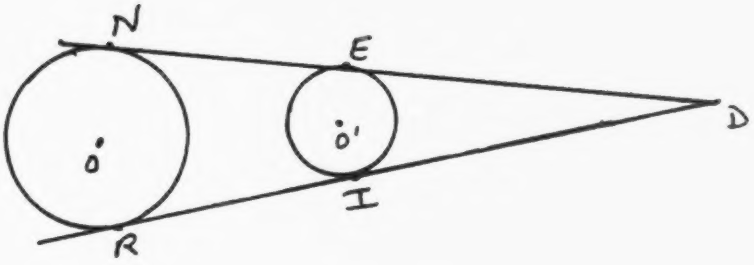
6.



SUMMARY:

1. What happens when a line is tangent to a circle?

2.



Why is $\overline{NE} \cong \overline{RI}$?

LESSONS: 50, 51

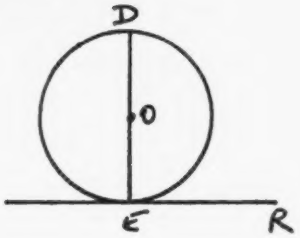
AIM: How do we measure angles formed by tan-chord, two tangents, tan-secant and two secants in a circle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the theorem "the measure of an angle formed by a tangent and a chord = $\frac{1}{2}$ the measure of its intercepted arc.
2. Apply the theorem "measure of an angle formed by 2 secants (2 tangents or a tan and sec) = $\frac{1}{2}$ difference of measure of the intercepted arcs".
3. Apply indicated theorems to algebraic and numeric problems.

VOCABULARY: tangent, secant, chord

MOTIVATION:



Consider the following:
Tangent $\overleftrightarrow{RE}$ intersects DE at point E on circle O
Find m ∢RED

Elicit:
If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.
Therefore, $\overline{OE} \perp \overline{RE}$
Perpendicular lines meet and form right angles m ∢RED = 90

DEVELOPMENT:

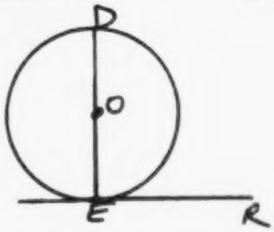
1. What conclusion can we draw?

   The measure of an angle formed by a tangent and the diameter (chord) is a right angle.

   Restate the above using arc $\overset{\frown}{DE}$.

The measure of an angle formed by a tangent and the diameter (chord) is = to $\frac{1}{2}$ the measure of its intercepted arc. $m \measuredangle RED = \frac{1}{2} \overset{\frown}{DE}$

2. Case I The chord is a diameter.



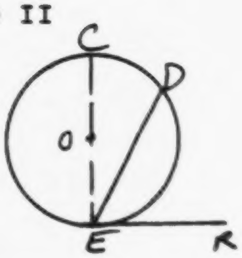
Given: $\overleftrightarrow{RE}$ is tangent to ⊙O at E
$\overline{DE}$ is a chord (diameter)

To Prove: $m \measuredangle RED = \frac{1}{2} \overset{\frown}{DE}$

** FORMAL PROOF IS OPTIONAL

Statements	Reasons
1. $\overline{DE}$ is a diameter of ⊙O and $\overleftrightarrow{RE}$ is tangent to O⊙ at E	1. Given
2. $\overline{DE} \perp \overline{RE}$ $m \measuredangle RED = 90$	2. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact. Perpendicular lines meet and form right angles.
3. $\overset{\frown}{DE}$ is a semicircle $m \overset{\frown}{DE} = 180$	3. A semicircle contains 180°
4. $m \measuredangle RED = \frac{1}{2} m \overset{\frown}{DE}$	4. Substitution

3. Case II Suppose Center O is outside ∡RED. (The chord is not a diameter.)



Given: $\overleftrightarrow{RE}$ is tangent to ⊙O at E
$\overline{DE}$ is a chord

To Prove: $m \measuredangle RED = \frac{1}{2} \overset{\frown}{DE}$

** FORMAL PROOF IS OPTIONAL

Informal Proof:

Draw diameter $\overline{CE}$.

$m\ \measuredangle CED = \frac{1}{2} m\ \overset{\frown}{CD}$

$m\ \measuredangle CER = m\ \measuredangle CED + m\ \measuredangle RED$

$m\ \overset{\frown}{EDC} = m\ \overset{\frown}{DE} + m\ \overset{\frown}{CD}$

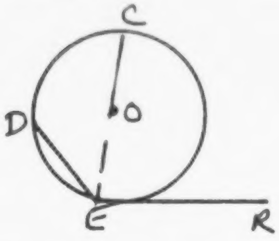
$m\ \measuredangle CER = \frac{1}{2}\ \overset{\frown}{EDC}$ Case I

$m\ \measuredangle CED + m\ \measuredangle RED = \frac{1}{2}(m\ \overset{\frown}{DE} + m\ \overset{\frown}{CD})$

$\frac{1}{2} m\ CD + m\ \measuredangle RED = \frac{1}{2} m\ \overset{\frown}{DE} + \frac{1}{2} m\ \overset{\frown}{CD}$

$m\ \measuredangle RED = \frac{1}{2}\ m\ \overset{\frown}{DE}$

4. Case III Suppose Center O is inside ∡RED.



Given: $\overleftrightarrow{RE}$ is tangent to ⊙O at E

$\overline{DE}$ is a chord

To Prove: $m\ \measuredangle RED = \frac{1}{2}\ \overset{\frown}{DE}$

** FORMAL PROOF IS OPTIONAL

Informal Proof:

Draw diameter $\overline{CE}$.

$m\ \measuredangle DEC = \frac{1}{2}\ \overset{\frown}{CD}$

$m\ \measuredangle RED = m\ \measuredangle DEC + m \measuredangle CER$

$m\ \overset{\frown}{DCE} = m\ \overset{\frown}{CD} + m\ \overset{\frown}{CE}$

$m\ \measuredangle CER = \frac{1}{2}\ \overset{\frown}{CE}$ Case I

$m\ \measuredangle RED = \frac{1}{2} m\ \overset{\frown}{CD} + \frac{1}{2} m\ \overset{\frown}{CE}$

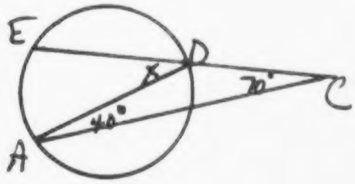
$m\ \measuredangle RED = \frac{1}{2}\ (m\ \overset{\frown}{CD} + m \overset{\frown}{CE})$

$m\ \measuredangle RED = \frac{1}{2}\ (\overset{\frown}{DE})$

5. Do Applications #1a,b,c.

6. Consider an angle formed by two secants.

Find m ∡ x



Using the exterior angle theorem.

$\measuredangle x° = 40° + 70°$

$\measuredangle x = 110°$

$40° = \measuredangle DAB$

$70° = \measuredangle ECA$

$x° = \measuredangle EDA$

Consider:

$\measuredangle x = \frac{1}{2}\overset{\frown}{EA}$ Inscribed Angle

$\measuredangle DAB = \frac{1}{2}\overset{\frown}{DB}$ Inscribed Angle

$\measuredangle EDA = \measuredangle DAB + \measuredangle ECA$

$\frac{1}{2}m\overset{\frown}{EA} = \frac{1}{2}m\overset{\frown}{DB} + \measuredangle ECA$

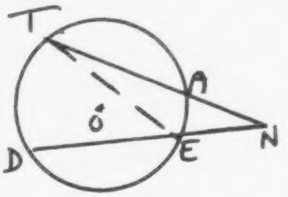
$\frac{1}{2}m\overset{\frown}{EA} - \frac{1}{2}m\overset{\frown}{DB} = \measuredangle ECA$

$\frac{1}{2}(m\overset{\frown}{EA} - m\overset{\frown}{DB}) = \measuredangle ECA$

What conclusion can we draw?

The measure of an angle formed by two secants is equal to one-half the difference of the intercepted arcs.

7. What seems to be the relationship between the measure of the angle formed by 2 secants and the intercepted arcs? Elicit theorem: The measure of an angle formed by 2 secants = $\frac{1}{2}$ difference of measures of the intercepted arcs.



Given: Secants $\overleftrightarrow{NAT}$ and $\overleftrightarrow{NED}$ intersect at N outside circle O

To Prove: $m\measuredangle N = \frac{1}{2}(m\overset{\frown}{TD} - m\overset{\frown}{AE})$

** FORMAL PROOF IS OPTIONAL

Informal Proof:

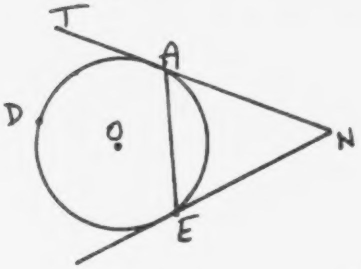
1. Draw chord $\overline{TE}$
2. $m\measuredangle N + m\measuredangle T = m\measuredangle TED$
3. $m\measuredangle N = m\measuredangle TED - m\measuredangle T$
4. $m\measuredangle TED = \frac{1}{2}m\overset{\frown}{TD}$
5. $m\measuredangle T = \frac{1}{2}m\overset{\frown}{AE}$

6. $m\angle N = \frac{1}{2} m\,\widehat{TD} - \frac{1}{2} m\,\widehat{AE}$

7. $m\angle N = \frac{1}{2} (m\,\widehat{TD} - m\,\widehat{AE})$

8. Do Application: #2a.

9. What seems to be the relationship between the angle formed by two tangents and its intercepted arcs? Elicit theorem: The measure of an angle formed by 2 tangents = $\frac{1}{2}$ the difference of measures of the intercepted arcs.



Given: Tangents $\overleftrightarrow{NAT}$ and $\overleftrightarrow{NE}$ intersect at N outside circle O

To Prove: $\angle N = \frac{1}{2}(m\,\widehat{ADE} - m\,\widehat{AE})$

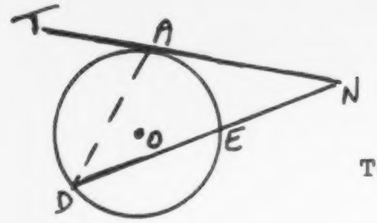
** FORMAL PROOF IS OPTIONAL

Informal Proof:

1. Draw chord $\overline{AE}$
2. $m\angle N + m\angle NEA = m\angle TAE$
3. $m\angle N = m\angle TAE - m\angle NEA$
4. $m\angle TAE = \frac{1}{2}$ ADE
5. $m\angle NEA = \frac{1}{2} m\,\widehat{AE}$
6. $m\angle N = \frac{1}{2} m\,\widehat{ADE} - \frac{1}{2} m\,\widehat{AE}$
7. $m\angle N = \frac{1}{2} (m\,\widehat{ADE} - m\,\widehat{AE})$

10. Do Application: 2B

11. What seems to be the relationship between the angle formed by a tangent and a secant and its intercepted arcs? Elicit theorem: The measure of an angle formed by a tangent and secant = $\frac{1}{2}$ difference of measures of the intercepted arcs.



Given: Tangents $\overleftrightarrow{NAT}$ and secant $\overleftrightarrow{NED}$ intersect at N outside circle O

To Prove: $m\angle N = \frac{1}{2}(m\widehat{AD} - m\widehat{AE})$

** FORMAL PROOF IS OPTIONAL

Informal Proof:

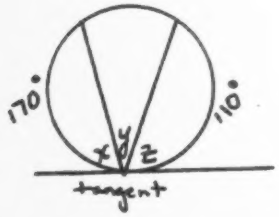
1. Draw chord $\overline{AD}$
2. $m\angle N + m\angle D = m\angle DAT$
3. $m\angle N = m\angle DAT - m\angle D$
4. $m\angle DAT = \frac{1}{2}\, m\,\widehat{AD}$
5. $m\angle D = \frac{1}{2}\, m\,\widehat{AE}$
6. $m\angle N = \frac{1}{2}\, m\,\widehat{AD} - \frac{1}{2}\, m\,\widehat{AE}$
7. $m\angle N = \frac{1}{2}\,(m\,\widehat{AD} - m\,\widehat{AE})$

12. Do Applications #2c, #3.

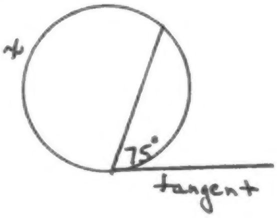
APPLICATIONS:

Find the values of x, y, z

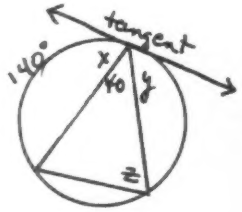
1. a)


170°
110°
x
y
z
tangent

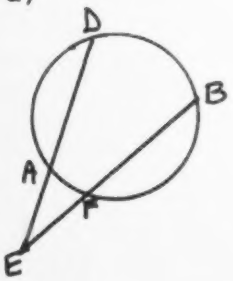
b)


x
75°
tangent

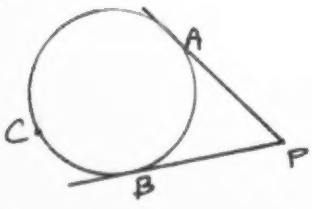
c)


tangent
140°
x
40
y
z

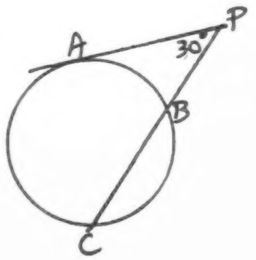
2. a)



b)



c)

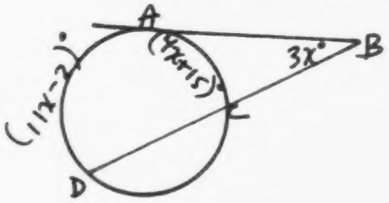


$m\widehat{AF} = 80$,
$m\widehat{DB} = 110$,
find $m\measuredangle E$.

If $m\measuredangle P = 100$
and $m\widehat{ACB} = 280$,
find $m\widehat{AB}$.

$\overline{PA}$ is a tangent,
$\overline{PC}$ is a secant. If
$m\widehat{AC} : m\widehat{AB} = 3:2$,
find $m\widehat{AC}$ and $m\widehat{AB}$.

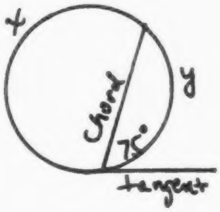
3. Find the values of x, $m\measuredangle B$, $m\widehat{AD}$, $m\widehat{AC}$.



SOLUTIONS TO APPLICATION PROBLEMS:

1.a)
$$\measuredangle x = \frac{1}{2}(170)$$
$$\measuredangle x = 85°$$
$$\measuredangle z = \frac{1}{2}(110)$$
$$\measuredangle z = 55°$$
$$\measuredangle y = \frac{1}{2}(360-280)$$
$$= \frac{1}{2}(180)$$
$$\measuredangle y = 40°$$

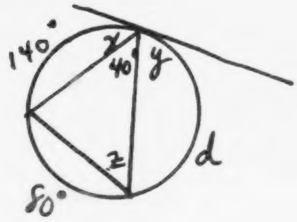
b)



$$75 = \frac{1}{2}(y)$$
$$150 = y$$
$$x + y = 360$$
$$x + 150 = 360$$
$$x = 210°$$

c) $\measuredangle z = \frac{1}{2}(140)° = 70°$ Inscribed Angle

$\measuredangle x = \frac{1}{2}(140)° = 70°$ Angle formed by tangent and chord.



$140 + 80 + d = 360$

$220 + d = 360$

$d = 140$

$\measuredangle y = \frac{1}{2}(140) = 70°$

Angle formed by tangent and chord

2. a) $\measuredangle E = \frac{1}{2}(110 - 80)$ angle formed by 2 secants

$= \frac{1}{2}(30)$

$= 15°$

b) $\measuredangle P = \frac{1}{2}(\overset{\frown}{ACB} - \overset{\frown}{AB})$ angle formed by 2 tangents

$100 = \frac{1}{2}(280 - \overset{\frown}{AB})$

$200 = 280 - \overset{\frown}{AB}$

$-80 = -\overset{\frown}{AB}$

$80° = \overset{\frown}{AB}$

c) $\measuredangle P = \frac{1}{2}(m\ \overset{\frown}{AC} - m\ \overset{\frown}{AB})$

$\measuredangle 30 = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{AB})$ angle formed by tangent and secant

$\measuredangle 30 = \frac{1}{2}(3x - 2x)$

$60 = 3x - 2x$

$60 = x$

$\overset{\frown}{AC} = 3x = 3(60) = 180°$

$\overset{\frown}{AB} = 2x = 2(60) = 120°$

3. Angle formed by tangent and secant

$\measuredangle B = \frac{1}{2}(m\ \overset{\frown}{AD} - m\ \overset{\frown}{AC})$

$3x = \frac{1}{2}((11x-2) - (4x+15))$

$3x = \frac{1}{2}(11x-2-4x-15)$

$6x = 11x-2-4x-15$

$6x = 7x-17$

$-x = -17$

$x = 17$

$$\measuredangle B = 3x = 3(17) = 51$$

$$\begin{aligned} m\,\overset{\frown}{AD} = 11x-2 &= 11(17)-2 \\ &= 187-2 = 185 \end{aligned}$$

$$\begin{aligned} m\,\overset{\frown}{AC} = 4x+15 &= 4(17)+15 \\ &= 68+15 \\ &= 83 \end{aligned}$$

SUMMARY:

1. Name three different angles formed by line segments intersecting outside a circle.
   a) How do we measure these angles?
   b) Include a sketch for each angle named.
   c) Give a sample exercise with its solution for each angle named.

2. How do we measure an angle formed by a chord and a tangent?
   a) Include a sketch.
   b) Give a sample exercise with its solution.

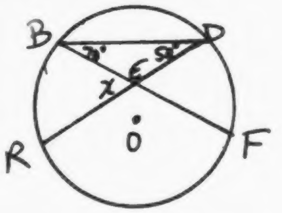
LESSON: 52

AIM: How do we measure an angle formed by two chords?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the theorem: The measure of an angle formed by 2 chords intersecting inside the circle = $\frac{1}{2}$ the sum of the measures of intercepted arcs.
2. Find the other arc when given the angle formed by 2 chords and one arc.
3. Apply angle measurement to complex problems.
4. Apply all angle measurement formulas to complex problems.

MOTIVATION:



Chord $\overline{DR}$ and chord $\overline{BF}$ intersect at E

Find $m\measuredangle x$

Using the exterior angle theorem.

$\measuredangle x = 70 + 50$

$\measuredangle x = 120°$

Let $50° = \measuredangle BDR$

Inscribed angle $\measuredangle BDR = \frac{1}{2}\, m\widehat{BR}$

Let $70° = \measuredangle DBF$

Inscribed angle $\measuredangle DBF = \frac{1}{2}\, m\widehat{DF}$

$\measuredangle x = \measuredangle DBF + \measuredangle BDR$

$\measuredangle x = \frac{1}{2}\, m\widehat{DF} + \frac{1}{2}\, m\widehat{BR}$

$\measuredangle x = \frac{1}{2}\,(m\widehat{DF} + m\widehat{BR})$

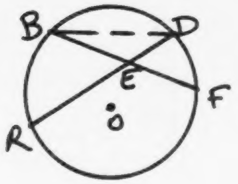
DEVELOPMENT:

1. What conclusion can we draw?

   The measure of an angle formed by 2 chords intersecting inside the circle = $\frac{1}{2}$ the sum of the measures of intercepted arcs.

2. ** FORMAL PROOF IS OPTIONAL

Consider the following diagram.



Given: Circle O with chords DR and BF intersecting at E.

To Prove: $m \measuredangle BER = \frac{1}{2}(m\ \overset{\frown}{BR} + m\ \overset{\frown}{DF})$

	Statements		Reasons
1.	Draw BD	1.	Two points determine a line.
2.	$m \measuredangle BER = \measuredangle DBE + \measuredangle BDR$	2.	The measure of an exterior angle of a triangle is = to the sum of the measures of the nonadjacent interior angles.
3.	$\measuredangle DBE = \frac{1}{2}\ m\overset{\frown}{DF}$ $\measuredangle BDR = \frac{1}{2}\ m\overset{\frown}{BR}$	3.	An inscribed angle is measured by $\frac{1}{2}$ its intercepted arc.
4.	$m \measuredangle BER = \frac{1}{2}\ m\ \overset{\frown}{DF} + \frac{1}{2}\ m\ \overset{\frown}{BR}$	4.	Substitution
5.	$m \measuredangle BER = \frac{1}{2}(m\ \overset{\frown}{BR} + m\ \overset{\frown}{DF})$	5.	Distributive Property

3. Do Applications #1a, #b, #c.

4. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If $m \measuredangle CEB = 50$ and arc $\overset{\frown}{CB} = 30°$, find the number of degrees in minor arc $\overset{\frown}{AD}$.

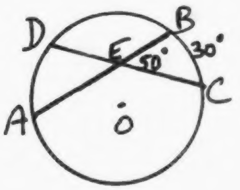
Draw the diagram:

$m \measuredangle CEB = \frac{1}{2}(m\ \overset{\frown}{CB} + m\ \overset{\frown}{AD})$

$50 = \frac{1}{2}(30 + x)$

$100 = 30 + x$

$70° = x$



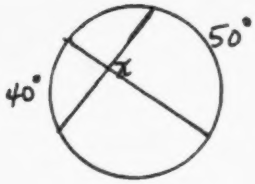
Let us apply angle measurement to more complicated problems.

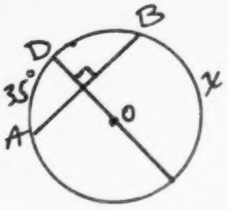
Do Applications #2, #3, #4.

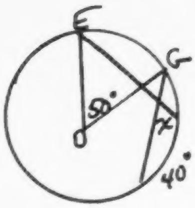
Do Applications * #5, * #6.

APPLICATIONS:

1. Find the values of x.







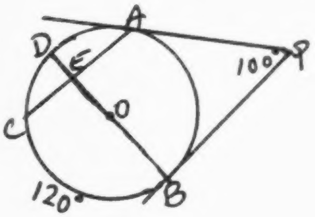
2. $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent to circle O at A and B respectively. Diameter $\overline{BD}$ and chord $\overline{AC}$ intersect at E, with m CB = 120 and m ∡APB = 100.

   Find m $\widehat{AB}$

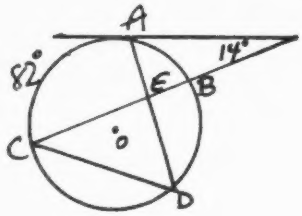
   Find m $\widehat{AD}$

   Find m ∡DEC

   Find m ∡PAC



3. $\overrightarrow{PA}$ is the tangent, $\overline{PEC}$ is a secant Chords $\overline{AD}$ and $\overline{BC}$ intersect at E, with $m\widehat{AC} = 82$, m∡P = 14 and $m\widehat{CD}:m\widehat{BD} = 5:3$ Find $m\widehat{AB}$, $m\widehat{BD}$ and $m\widehat{CD}$.

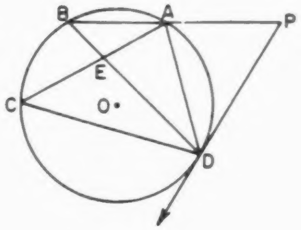


4. June 1985 Regents.

In the accompanying diagram, *B* is the midpoint of $\widehat{AC}$, triangle *ADC* is inscribed in circle *O*, chords $\overline{AC}$ and $\overline{BD}$ intersect at *E*, $\overrightarrow{PD}$ is tangent to circle *O* at *D*, $\overline{PAB}$ is a secant, and $m\widehat{BA}:m\widehat{AD}:m\widehat{DC} = 2:3:5$.

Find:

a. $m\widehat{BC}$ [2]
b. m∠*ADC* [2]
c. m∠*AEB* [2]
d. m∠*ADP* [2]
e. m∠*P* [2]

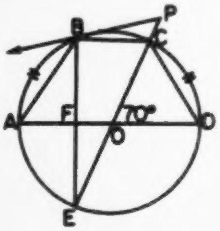


* 5. June 1983 Regents

In the accompanying diagram, *ABCD* is inscribed in circle *O*, m∠*COD* = 70, $\overline{AOD}$ is a diameter, $\overrightarrow{PB}$ is tangent to circle *O* at *B*, $\overline{PCOE}$ is a secant, and m$\overset{\frown}{BA}$ = m$\overset{\frown}{CD}$.

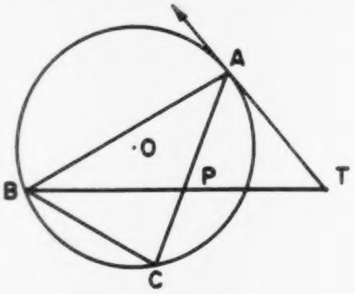
Find:

a m$\overset{\frown}{BC}$ [2]
b m∠*PBC* [2]
c m∠*A* [2]
d m∠*BPE* [2]
e m∠*AFE* [2]



* 6. January 1984 Regents

In the accompanying figure, △*ABC* is inscribed in circle *O*, $\overline{BT}$ bisects ∠*CBA*, $\overrightarrow{TA}$ is tangent to circle *O*, and m∠*BAC*:m∠*CBA*:m∠*ACB* = 2:3:4.



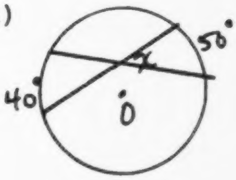
Find:

a m∠*BAC* [2]
b m$\overset{\frown}{BC}$ [2]
c m∠*CPT* [2]
d m∠*PAT* [2]
e m∠*T* [2]

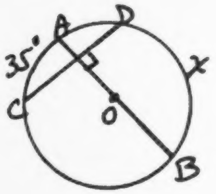
SOLUTIONS TO APPLICATION PROBLEMS:

1. Find the values of x.

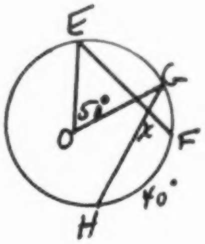
a)



b)



c)



a)

$\measuredangle x = \frac{1}{2}(50+40)$

$\measuredangle x = \frac{1}{2}(90)$

$\measuredangle x = 45^\circ$

b)

$\overset{\frown}{AD} = 35^\circ$

A diameter $\perp$ to a chord bisects the chord and both pairs of arcs.

$\overset{\frown}{AD}+\overset{\frown}{DB} = 180^\circ$

$35+x = 180$

$x = 145^\circ$

OR

$90 = \frac{1}{2}(35+x)$

$180 = 35+x$

$145^\circ = x$

c)

$\overset{\frown}{EG} = 50^\circ$

A central angle is measured by its intercepted arc.

$\measuredangle x = \frac{1}{2}(50+40)$

$\measuredangle x = \frac{1}{2}(90)$

$\measuredangle x = 45^\circ$

2. a) Angle formed by 2 tangents.

$\measuredangle P = \frac{1}{2}((180+x)-(180-x))$

$100 = \frac{1}{2}(180+x-180+x)$

$100 = \frac{1}{2}(2x)$

$100^\circ = x$

$\overset{\frown}{AB} = 180-x$

$\overset{\frown}{AB} = 180-100 = 80^\circ$

Note: $\overline{CD}$ is a diameter

$\overset{\frown}{CD}+\overset{\frown}{CB} = 180^\circ$

$\overset{\frown}{CD}+120 = 180$

$\overset{\frown}{CD} = 60^\circ$

Let $\overset{\frown}{AD} = x$

$\overset{\frown}{AB} = 180-x$

b) $m\,\overset{\frown}{AD} = x = 100$

c) Angle formed by 2 chords.

$m\,\measuredangle DEC = \frac{1}{2}(60+80)$

$= \frac{1}{2}(140)$

$m\,\measuredangle DEC = 70$

d) Angle formed by tangent and chord.

$m\,\measuredangle PAC = \frac{1}{2}\overset{\frown}{ABC}$

$= \frac{1}{2}(200)$

$m\,\measuredangle PAC = 100^\circ$

Note: $\overset{\frown}{ABC} = \overset{\frown}{AB}+\overset{\frown}{BC}$

$\overset{\frown}{ABC} = 80+120 = 200^\circ$

3. Angle formed by tangent and secant

$\measuredangle P = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{AB})$

$14 = \frac{1}{2}(82 - \overset{\frown}{AB})$

$28 = 82 - \overset{\frown}{AB}$

$\overset{\frown}{AB} = 54°$

$\overset{\frown}{AC} + \overset{\frown}{AB} + \overset{\frown}{CD} + \overset{\frown}{BD} = 360°$

$82 + 54 + 5x + 3x = 360$

$136 + 8x = 360$

$8x = 224$

$x = 28°$

$\overset{\frown}{BD} = 3x = 3(28) = 84°$

$\overset{\frown}{CD} = 5x = 5(28) = 140°$

4. a) B is midpoint of $\overset{\frown}{AC}$

Therefore $m\overset{\frown}{BC} = m\overset{\frown}{BA}$

$m\overset{\frown}{BA} + m\overset{\frown}{BC} + m\overset{\frown}{AD} + m\overset{\frown}{DC} = 360$

$2x + 2x + 3x + 5x = 360$

$12x = 360$

$x = 30°$

$m\overset{\frown}{BA} = 2(30) = 60$

$m\overset{\frown}{AD} = 3(30) = 90$

$m\overset{\frown}{DC} = 5(30) = 150$

$m\overset{\frown}{BC} = 2x = 2(30) = 60$

b) Inscribed angle

$m\measuredangle ADC = \frac{1}{2}m\,\overset{\frown}{CBA}$

$m\,\overset{\frown}{CBA} = m\,\overset{\frown}{BC} + m\,\overset{\frown}{BA}$

$m\,\overset{\frown}{CBA} = 60 + 60 = 120$

$m\measuredangle ADC = \frac{1}{2}(120)$

$m\measuredangle ADC = 60$

c) Angle formed by 2 chords

$m\measuredangle AEB = \frac{1}{2}(m\,\overset{\frown}{BA} + m\,\overset{\frown}{DC})$

$= \frac{1}{2}(60 + 150)$

$m\measuredangle AEB = \frac{1}{2}(210) = 105$

d) Angle formed by tangent and chord.

$m\measuredangle ADP = \frac{1}{2}\overset{\frown}{AD}$

$m\measuredangle ADP = \frac{1}{2}(90) = 45$

e) Angle formed by tangent and secant.

$m\measuredangle P = \frac{1}{2}(m\overset{\frown}{BCD} - m\overset{\frown}{AD})$ *Where* $m\overset{\frown}{BCD} = m\overset{\frown}{BC} + m\overset{\frown}{DC}$

$= 60 + 150$

$m\overset{\frown}{BCD} = 210$

$$m\measuredangle P = \frac{1}{2}(m\overset{\frown}{BCD} - m\overset{\frown}{AD})$$
$$= \frac{1}{2}(210 - 90)$$
$$m\measuredangle P = \frac{1}{2}(120) = 60$$

5. a) $\overline{AOD}$ is diameter

$\measuredangle COD = \overset{\frown}{CD} = 70°$ Central angle

$m\overset{\frown}{BA} = m\overset{\frown}{CD} = 70$ Given

$$m\overset{\frown}{BA} + m\overset{\frown}{CD} + m\overset{\frown}{BC} = 180$$
$$70 + 70 + m\overset{\frown}{BC} = 180$$
$$140 + m\overset{\frown}{BC} = 180$$
$$m\overset{\frown}{BC} = 40$$

b) Angle formed by tangent and chord.

$$m\measuredangle PBC = \frac{1}{2}\overset{\frown}{BC}$$
$$m\measuredangle PBC = \frac{1}{2}(40) = 20$$

c) Inscribed angle

$m\measuredangle A = \frac{1}{2}m\overset{\frown}{BCD}$ But $m\overset{\frown}{BCD} = m\overset{\frown}{BC} + m\overset{\frown}{CD}$

$m\measuredangle A = \frac{1}{2}(110)$ $= 40 + 70$

$= 55$ $= 110$

d) Angle formed by tangent and secant.

$m\measuredangle BPE = \frac{1}{2}(m\overset{\frown}{BAE} - m\overset{\frown}{BC})$ Note: $m\measuredangle AOE = m\measuredangle COD$

Vertical angles are congruent

$m\measuredangle AOE = m\overset{\frown}{AE}$

$70 = m\overset{\frown}{AE}$

$$m\overset{\frown}{BAE} = m\overset{\frown}{BA} + m\overset{\frown}{AE}$$
$$= 70 + 70$$
$$= 140$$

$$m\measuredangle BPE = \frac{1}{2}(m\overset{\frown}{BAE} - m\overset{\frown}{BC})$$
$$= \frac{1}{2}(140 - 40)$$
$$\frac{1}{2}(100) = 50$$

e) Angle formed by 2 chords.

$m\angle AFE = \frac{1}{2}(m\overset{\frown}{AE} + m\overset{\frown}{BCD})$

Note: $m\overset{\frown}{BCD} = m\overset{\frown}{BC} + m\overset{\frown}{CD}$

$= 40 + 70$

$m\overset{\frown}{BCD} = 110$

$m\angle ARD = \frac{1}{2}(70 + 110)$

$= \frac{1}{2}(180)$

$m\angle AFE = 90$

6. a) $m\angle BAC : m\angle CBA : m\angle ACB$

$2x + 3x + 4x = 180$

$9x = 180$

$x = 20$

$m\angle BAC = 2x = 2(20) = 40$

b) Inscribed angle

$m\angle BAC = \frac{1}{2}m\overset{\frown}{BC}$

$40 = \frac{1}{2}m\overset{\frown}{BC}$

$80 = m\overset{\frown}{BC}$

c) $\overline{BT}$ bisects $\angle CBA$ Therefore $\angle ABP = \frac{1}{2}\angle CBA$

$\angle ABP = \frac{1}{2}(60) = 30°$

Sum of angles of triangle = 180

$\angle ABP + \angle BAC + \angle APB = 180$

$30 + 40 + y = 180$

$70 + y = 180$

$y = 110$

Vertical angles are congruent.

$m\angle CPT \cong m\angle APB$

$m\angle CPT = 110$

d) Inscribed angle

$$m\measuredangle BAC = \frac{1}{2}\overset{\frown}{AC}$$
$$40 = \frac{1}{2}\overset{\frown}{AC}$$
$$80^\circ = \overset{\frown}{AC}$$

angle formed by tangent and chord

$$m\measuredangle PAT = \frac{1}{2}\overset{\frown}{AC}$$
$$m\measuredangle PAT = \frac{1}{2}(80) = 40$$

e) Supplementary angles

$$m\measuredangle CPT + m\measuredangle TPA = 180$$
$$110 + \measuredangle TPA = 180$$
$$\measuredangle TPA = 70^\circ$$

Sum of angles of triangle = 180°

$$m\measuredangle PAT + m\measuredangle TPA + m\measuredangle T = 180$$
$$40 + 70 + \measuredangle T = 180$$
$$110 + \measuredangle T = 180$$
$$\measuredangle T = 70^\circ$$

SUMMARY:

How do we measure an angle whose vertex is at the center of the circle?

How do we measure an angle whose vertex is on the circle?

How do we measure an angle formed by two intersecting chords inside the circle?

How do we measure an angle formed by a chord and a tangent?

How do we measure an angle formed by two secants, or two tangents or a tangent and a secant?

LESSONS: 53, 54

AIM: How do we find the length of line segments?

PERFORMANCE OBJECTIVES: The student will be able to...

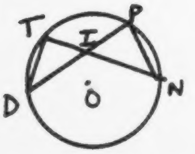
1. Prove informally the theorems.
    a) If two chords intersect within a circle, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other.
    b) If two secants are drawn to a circle from the same external point, then the product of the lengths of one secant and its external segment equals the product of the lengths of the other secant and its external segment.
2. Apply the above to algebraic exercises.

VOCABULARY: similar polygons, similar triangles, corresponding sides, secant, external segment of the secant

MOTIVATION:

1. Pose the questions: A. How do we prove 2 triangles similar?
    Ans. aa = aa
    B. What are the properties of similar polygons?
    Ans. 1) All pairs of corresponding angles are $\cong$.
    2) The ratios of the lengths of all pairs of corresponding sides are equal.

2. Prove $\triangle DIP \sim \triangle TIN$



Informal Proof:

$\angle T \cong \angle P$ Why? Inscribed angles that intercept the same arc are $\cong$.

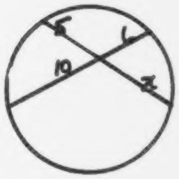
$\angle D \cong \angle N$ Why? Inscribed angles that intercept the same arc are $\cong$.

$\triangle DIP \sim \triangle TIN$ Why? aa $\cong$ aa

What conclusions can we make? Corresponding sides of similar s are in proportion. Therefore, $\frac{\overline{DI}}{\overline{NI}} = \frac{\overline{TI}}{\overline{PI}}$

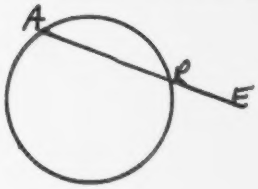
DEVELOPMENT:

1. Remember the theorem. "In a proportion, the product of the means is equal to the product of the extremes." Therefore $(\overline{NI})(\overline{TI}) = (\overline{DI})(\overline{PI})$
   Can we generalize the above in an algebraic exercise?



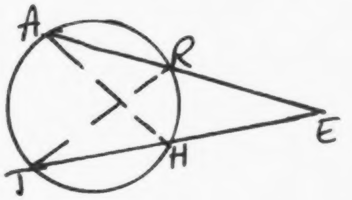
$5x = (6)(10)$
$5x = 60$
$x = 12$

2. Let us generalize the above in words: Elicit: "If two chords intersect within a circle, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other."
3. Do Applications #1, #2a, #3a.
4. Does a similar relationship exist for two secants intersecting outside the circle?



$\overline{AR}$ is a chord
$\overline{RE}$ is an external segment of the secant
$\overline{AE}$ is a secant

Prove informally the theorem:



If two secants are drawn to a circle from the same external point, then the product of the lengths of one secant and its external segment equals the product of the lengths of the other secant and its external segment.

Plan: Draw $\overline{TR}$ and $\overline{AH}$

$\angle A \cong \angle T$ Why? Inscribed angles that intercept the same are are $\cong$.

$\angle E \cong \angle E$ Why? Identity

$\triangle RET \sim \triangle HEA$ Why? aa = aa

$\frac{\overline{AE}}{\overline{TE}} = \frac{\overline{RE}}{\overline{RE}}$ Why? Corresponding sides of similar triangles are in proportion.

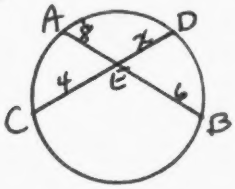
Therefore $(\overline{AE})(\overline{RE}) = (\overline{TE})(\overline{HE})$ Why? In a proportion, the product of the means is equal to the product of the extremes.

5. Do Applications #2B, #3B.

APPLICATIONS:

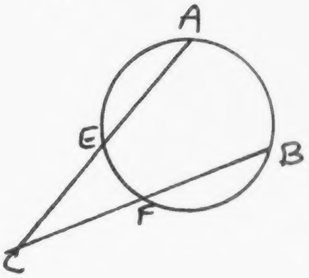
1. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If AE = 6, BE = 12, and CE = 8, find the length of $\overline{DE}$.

2. a.



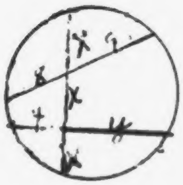
Find the length of $\overline{ED}$

b.



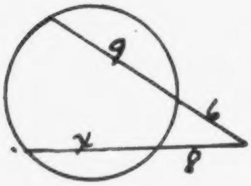
CA = 6, EC = 4, CB = 8 Find the length of $\overline{CF}$

3. a.



Find the length of
x and y

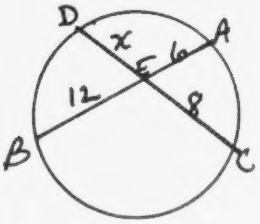
b.



Find the length of x

SOLUTIONS TO APPLICATION PROBLEMS:

1.


D
x
E
6
A
12
8
B
C

$(\overline{AE})(\overline{BE}) = (\overline{CE})(\overline{DE})$

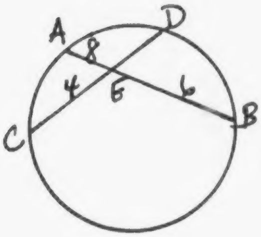
$(6)(12) = 8x$

$72 = 8x$

$9 = x$

$DE = 9$

2. a.



$(\overline{AE})(\overline{BE}) = (\overline{CE})(\overline{DE})$

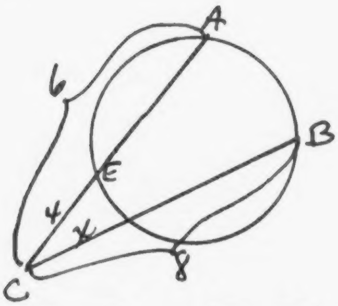
$(8)(6) = 4x$

$48 = 4x$

$12 = x$

$ED = 12$

b.



$(\overline{AC})(\overline{CE}) = (\overline{CB})(\overline{CF})$

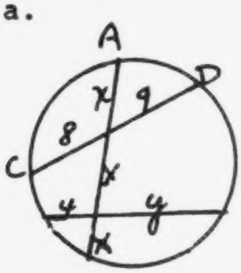
$(6)(4) = 8x$

$24 = 8x$

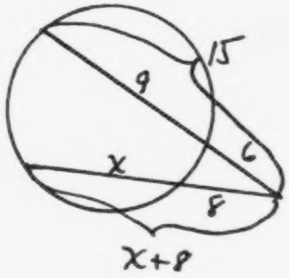
$3 = x$

$CF = 3$

3. a.



b.



$(\overline{CE})(\overline{ED}) = (\overline{AE})(\overline{EB})$

$(8)(9) = (x)(2x)$

$72 = 2x^2$

$36 = x^2$

$\pm 6 = x$

length is positive

$x = 6$

$(\overline{GF})(\overline{FH}) = (\overline{AF})(\overline{FB})$

$4y = (12)(6)$

$4y = 72$

$y = 18$

$(15)(6) = (8+x)(8)$

$90 = 64+8x$

$26 = 8x$

$3\frac{1}{4} = x$

SUMMARY:

State the properties of similar triangles.

State the relationship that exists if two chords intersect within a circle.

State the relationship that exists if two secants are drawn to a circle from the same external point.

LESSON: 55

AIM: How do we find the length of line segments?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove informally the theorem:
   a) When a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the whole secant and its external segment.
2. Review: Tangents drawn to a circle from the same external points are $\cong$.
3. Apply the above to algebraic exercises.

VOCABULARY: tangent, secant, mean proportional, external segment of secant

MOTIVATION:

If secant $\overline{PB}$ is rotated clockwise about P, what happens?

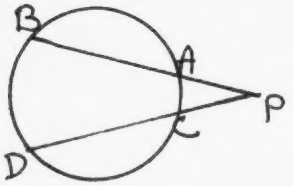


Diagram 1

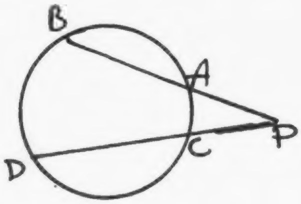


Diagram 2

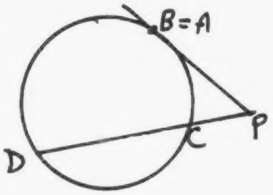


Diagram 3

If secant $\overline{PB}$ in the above diagram is rotated clockwise about P, A and B get closer together. A and B will eventually coincide. Then $\overline{PA} \cong \overline{PB}$ and $\overline{PB}$ is a tangent to the circle.

DEVELOPMENT:

1. Considering two secants $\overline{PB}$ and $\overline{PD}$ we know

$$(\overline{PB})(\overline{PA}) = (\overline{PD})(\overline{PC})$$

When A coincides with B, we can conclude

$$(\overline{PA})(\overline{PA}) = (\overline{PD})(\overline{PC})$$

$$(\overline{PA})^2 = (\overline{PD})(\overline{PC})$$

Ask the students to generalize the above in words:

When a tangent and a secant are drawn to a circle, the square of length of the tangent is equal to the product of the lengths of the secant and its external segment.

2. Consider $(\overline{PA})^2 = (\overline{PD})(\overline{PC})$

Let us write the above in terms of a proportion.

$$\frac{\overline{PD}}{\overline{PA}} = \frac{\overline{PA}}{\overline{PC}}$$

Thus $\overline{PA}$ is the mean proportional between $\overline{PD}$ and $\overline{PC}$.

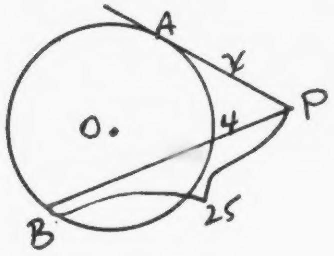
What conclusion can we make?

When a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the whole secant and its external segment.

3. $\overline{PA}$ is tangent to circle O.

$\overline{PB}$ is a secant

Find the value of x.



$$\frac{25}{x} = \frac{x}{4}$$

$$x^2 = 100$$

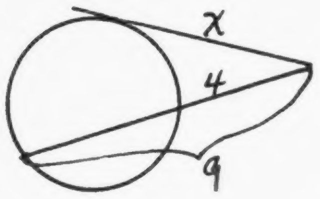
$$x = \pm 10$$

Length is positive.

$$x = 10$$

4. Do Application #1.

5. If a secant to a circle from an external point is 9 and its external segment is 4, what is the length of the tangent from that point.



$$\frac{9}{x} = \frac{x}{4}$$

$$x^2 = 36$$

$$x = \pm 6$$

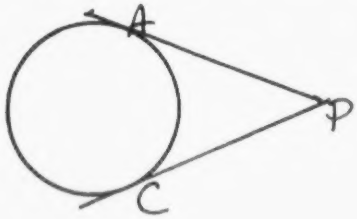
Length is positive.

$$x = 6$$

6. Do Application #2.

7. Do Applications #3A, #3B.

8. Consider two tangents drawn from an external point. What conclusion can we make?

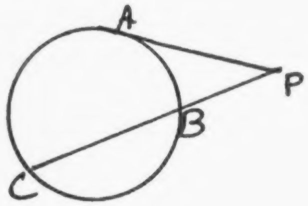


Recall: Two tangents to a circle from the same external point are $\cong$.

9. Do Applications #3C, 4

APPLICATIONS:

1.

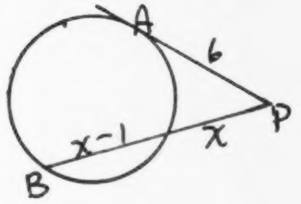


$\overline{PA}$ is a tangent, $\overline{PC}$ is a secant.

If CB = 5, BP = 4 find $\overline{PA}$.

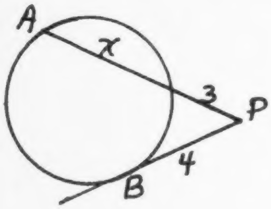
2. From point P outside a circle, tangent $\overline{PD}$ and secant $\overline{PA}$ are drawn. Secant $\overline{PA}$ intersects the circle at C. PD = 8 and the secant is four times its external segment. Find the secant.

3. a) $\overline{PA}$ is a tangent
$\overline{PB}$ is a secant



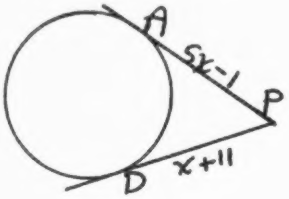
Find the value of x

b) $\overline{PA}$ is a secant
$\overline{PB}$ is a tangent



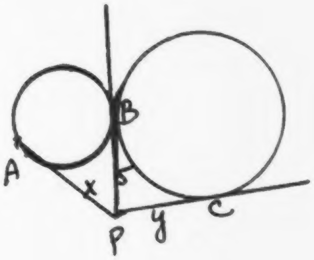
Find the value of x

c) $\overline{PA}$ and $\overline{PD}$ are tangents



Find $\overline{PA}$

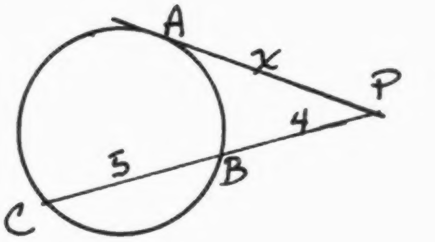
4. $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ are tangents



Find the values of x and y.

SOLUTIONS TO APPLICATION PROBLEMS:

1.



$$\frac{\overline{PC}}{\overline{PA}} = \frac{\overline{PA}}{\overline{PB}}$$

$$\frac{9}{x} = \frac{x}{4}$$

$$x^2 = 36$$

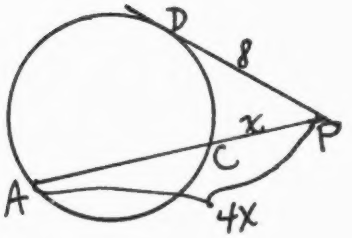
$$x = \pm 6$$

Length is positive.

$$x = 6$$

Therefore PA = 6

2.



$$\frac{4x}{8} = \frac{8}{x}$$

$$4x^2 = 64$$

$$x^2 = 16$$

$$x = \pm 4$$

Length is positive.

$$x = 4$$

3. a) $\frac{2x-1}{6} = \frac{6}{x}$

$$(x)(2x-1) = (6)(6)$$

$$2x^2-x = 36$$

$$2x^2-x-36 = 0$$

$$(2x-9)(x+4) = 0$$

$$2x-9 = 0 \text{ or } x+4 = 0$$

$$2x = 9 \qquad x = -4$$

$$x = \frac{9}{2} \qquad \text{reject}$$

$$\text{or } 4\tfrac{1}{2}$$

b) $\frac{3+x}{4} = \frac{4}{3}$

$$3(3+x) = 16$$

$$9+3x = 16$$

$$3x = 7$$

$$x = \frac{7}{3}$$

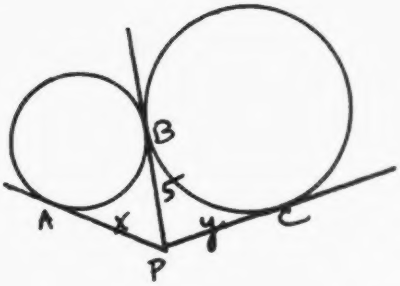
c) $5x-1 = x+11$

$4x = 12$

$x = 3$

$PA = 5x-1 = 5(3)-1 = 14$

4.



Tangents drawn to a circle from the same external points are =.

$x = 5$

$y = 5$

Find x ; Find y

SUMMARY:

What relationship exists when a tangent and a secant are drawn to a circle from an external point?

What relationship exists when two tangents are drawn to a circle from the same external point?

LESSONS: 56, 57

AIM: How can we find the circumference, area, length of arc, area of sector and area of a segment of a circle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define circumference, area, sector and segment of a circle.
2. State the formula for circumference of a circle.
3. Find the length of an arc of a circle.
4. State the formula for the area of a circle.
5. Find the area of a sector and area of a segment of a circle.
6. Apply to algebraic and numeric exercises.

VOCABULARY: circumference, area of a circle, sector of a circle, segment of a circle

MOTIVATION:

EXPERIMENT:

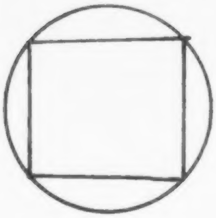
1) Using a string, measure the circumference of a tin can. Remember: The circumference of a circle is the perimeter of the circle or the distance around the circle.

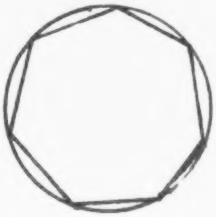
2) Using the string, measure the diameter of the circle.

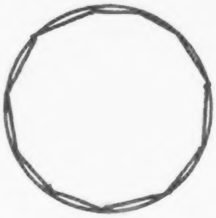
3) Use several different sized tin cans. Measure the circumferences. Students should observe that in every case the circumference is slightly more than three times the diameter.
The ratio $\frac{C}{D}$ is a constant which is an irrational number approximately equal to 3.1415926536.
The name commonly used to refer to the number is "pi", the name of the Greek letter π. A rational number close to the value of π is $3\frac{1}{7}$ or $\frac{22}{7}$ which has the decimal value $3.142857\overline{142857}$. $\frac{22}{7}$ and 3.14 are commonly used as approximations of the value of π.

4. Another Demonstration:

Consider a sequence of regular polygons inscribed in a circle as shown below.







Ask: "What conclusions can be drawn if the number of sides of the regular polygons continues to increase?" Elicit: The circumference of the circle is the limit of the perimeter of these inscribed polygons as the number of sides increases. These perimeters will get closer and closer to $2(\pi)(r)$ or $2\pi r$.

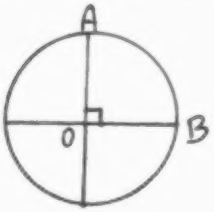
DEVELOPMENT:

1. The circumference C of a circle with diameter 'd' and radius 'r' is given by the formulas.

   $C = \pi d$ and $C = 2\pi r$

2. Do Applications 1a,b,c,d
   2a,b,c,d
   3a,b
   4a,b.

3. Find the length of (minor) arc AB if OB = 6 where O is the center of the circle.



$m\angle AOB = m\overset{\frown}{AB}$

$\angle AOB = 90°$

$\overset{\frown}{AB} = 90°$

Ask "How can we find the length of an arc?" Elicit: To find the length of an arc of a circle, you need to know the circumference of the circle and the fractional part of the circle contained in the arc.

For example $\frac{90°}{360}$ or $\frac{1}{4}$ of the circumference.

$$\text{Length of arc} = \frac{90}{360} \cdot C$$

$$\frac{1}{4} \cdot 2\pi r$$

$$\frac{1}{4}\, 2(\pi)(6) = \frac{12\pi}{4} = 3\pi$$

Ask the students to generalize the above in words:

$$\frac{\text{degree measure of arc}}{360°} = \frac{\text{length of arc}}{\text{circumference}}$$

4. Do Applications #5, #6.

5. Consider a regular n-sided polygon with sides of length "s", circumscribed about circle O.

   Ask: "How can we find the area of this n-sided regular polygon?"

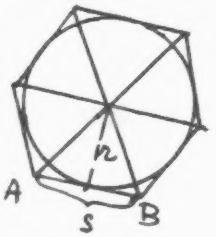
   Elicit: The Area of $\triangle$ AOB $= \frac{1}{2}rs$

   Thus the Area of the polygon is

   $\text{Area} = n \cdot \frac{1}{2}rs$

   $ns = \text{Perimeter} = P$

   $\text{Area} = \frac{1}{2}rP$



   Ask: "What happens to the perimeter of the polygon as the number of sides increases?"

   Elicit: As the number of sides of polygon increases, the perimeter of the polygon approaches the circumference of the circle and the area of the polygon approaches the area of the circle.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}rC \\ &= \tfrac{1}{2}r(2\pi r) \\ &= \pi r^2 \end{aligned}$$

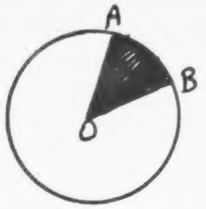
   Therefore, Area of circle with radius r.

$$A = \pi r^2$$

6. Do Applications #7, #8, #9, #10.

Ask the class to explain how the shaded region below is formed. Elicit the description and explain to the class that this shaded region is called a sector of a circle.

7. Define: A **SECTOR** of a circle is a part of the interior of the circle bounded by two radii and an arc.



Given: $\measuredangle AOB = 40°$
$OB = 4$

How can we find the area of a sector of a circle with radius 4? Elicit:

Since 40° is $\frac{40}{360}$ or $\frac{1}{9}$ of 360°, it follows that the area of the sector is $\frac{1}{9}$ of the area of the whole circle.

Area of circle $= \pi r^2 = \pi(4)(4) = 16\pi$

Area of the sector $= \frac{1}{9}(16\pi) = \frac{16}{9}\pi$

Elicit the generalization

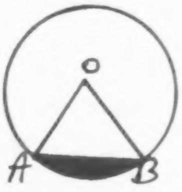
$$\text{Area of SECTOR AOB} = \frac{n}{360} \cdot \pi r^2$$

Area A of a SECTOR with an arc of measure n in a circle with radius r is

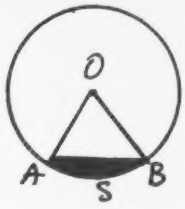
$$A = \frac{n}{360} \cdot \pi r^2$$

8. Do Applications #11, #12.

Have the class explain how the shaded region below is formed. Elicit the description and explain that this shaded region is called a segment of a circle.



9. Define: A **SEGMENT** of a circle is the figure formed by an arc of the circle and its chord.



Ask: "How can we find the area of segment ASB?"
Elicit: In circle O,
area of segment ASB = area of sector OASB - area of triangle AOB.

10. Do Application #13.

APPLICATIONS:

1. Find the circumference of circles with these radii.
   a) 5 b) 7 c) 9 d) 2
2. Find the circumference of circles with these diameters.
   a) 12 b) 3 c) 7.5 d) $\frac{1}{2}$
3. Find radius of circle with the given circumference.
   a) 6π b) $12\sqrt{2}\pi$
4. Find the diameter of a circle with the given circumference.
   a) π b) 16π
5. Find the length of an arc of measure 50° in a circle with diameter 4.
6. The length of an arc is 4π. If the degree measure of an arc is 60, find radius of the circle.
7. Find the area of a circle with radius 6.
8. Find the area of a circle with diameter 7.
9. Find the diameter of a circle with given area of 16π.
10. Find the circumference of a circle whose area is 121π.
11. Find the area of a sector of a circle, if the central angle of the sector has measure 150 and radius 5.

12. The area of a sector of a circle is 18π. If the measure of the central angle of the sector is 80, find the radius of the circle.

13. In a circle whose radius is 10, find the area of a minor segment whose arc has a central angle of 60°. (Answer may be left in radical form and in terms of π.)

SOLUTIONS TO APPLICATION PROBLEMS:

1. $C = 2\pi r$

a) $C = 2\pi(5)$
$C = 10\pi$

b) $C = 2\pi(7)$
$C = 14\pi$

c) $C = 2\pi r$
$C = 2\pi(9)$
$C = 18\pi$

d) $C = 2\pi r$
$C = 2\pi(2)$
$C = 4\pi$

2. $C = \pi d$

a) $C = \pi(12)$
$C = 12\pi$

b) $C = \pi d$
$C = \pi 3$
$C = 3\pi$

c) $C = \pi d$
$C = \pi(7.5)$
$C = 7.5\pi$

d) $C = \pi d$
$C = \pi \frac{1}{2}$
$C = \frac{1}{2}\pi$

3. a) $C = 2\pi r$
$6\pi = 2\pi r$
$3 = r$

b) $C = 2\pi r$
$12\sqrt{2}\pi = 2\pi r$
$6\sqrt{2} = r$

4. a) $C = \pi d$
$\pi = \pi d$
$1 = d$

b) $C = \pi d$
$16\pi = \pi d$
$16 = d$

5. $\frac{\text{degree measure arc}}{360°} = \frac{\text{length or arc}}{\text{circumference}}$

$$\frac{50}{360} = \frac{x}{4\pi}$$
$$360x = 200\pi$$
$$x = \frac{200\pi}{360}$$
$$x = \frac{5}{9}\pi$$

6. $\frac{n}{360} = \frac{\text{length of arc}}{\text{circumference}}$

$\frac{60}{360} = \frac{4\pi}{2\pi r}$

$\frac{1}{6} = \frac{2}{r}$

$r = 12$

7. Area $= \pi r^2$

$= \pi(6)(6)$

$= 36\pi$

8. Area $= \pi r^2$; diameter = 7, radius $= \frac{7}{2}$

$= \pi\left(\frac{7}{2}\right)\left(\frac{7}{2}\right)$

$= \frac{49}{4}\pi$

9. Area $= \pi r^2$

$16\pi = \pi r^2$

$16 = r^2$

$\pm 4 = r$ diameter = 8

length is positive, $r = 4$ diameter = 8

10. Area $= \pi r^2$; Circumference $= 2\pi r$

$121\pi = \pi r^2$ $= 2\pi(11)$

$121 = r^2$ $= 22\pi$

$\pm 11 = r$

length is positive, $r = 11$

11. $\frac{n}{360} = \frac{\text{area of sector}}{\text{area of circle}}$

$\frac{150}{360} = \frac{x}{\pi r^2}$

$\frac{150}{360} = \frac{x}{\pi(5)(5)}$

$\frac{150}{360} = \frac{x}{25\pi}$

$\frac{5}{12} = \frac{x}{25\pi}$

$12x = 125\pi$

$x = \frac{125\pi}{12}$

12. $\frac{n}{360} = \frac{\text{area of sector}}{\text{area of circle}}$

$\frac{80}{360} = \frac{18\pi}{\pi r^2}$

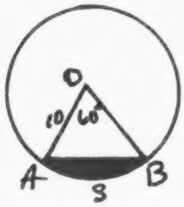
$80r^2 = (360)(18)$

$80r^2 = 6480$

$r^2 = 81$

$r = 9$

13.



Area of sector OASB $= \frac{n}{360} \cdot \pi r^2$

$= \frac{60}{360} \pi(10)(10)$

$= \frac{1}{6} (100\pi) = \frac{50\pi}{3}$

Area of equilateral triangle AOB $= \frac{s^2\sqrt{3}}{4}$

$= \frac{10^2\sqrt{3}}{4}$

$= \frac{100\sqrt{3}}{4} = 25\sqrt{3}$

Area of segment ASB = area of sector OASB - area of $\triangle$ AOB

Area of segment ASB $= \frac{50\pi}{3} - 25\sqrt{3}$

SUMMARY:

1. What is the relationship between the diameter of a circle and its circumference?
2. What is the relationship between the radius of a circle and its circumference?
3. What is π?
4. Explain how to find each of the following:
   Length of an arc of a circle.
   Area of a sector of a circle.
   Area of segment of a circle.

LESSONS: 58, 59

AIM: How do we construct the circumscribed and inscribed circle of a given polygon and construct the tangent from a given point on or outside of a circle?
(What are the constructions related to the circle?)

PERFORMANCE OBJECTIVES: The student will be able to...

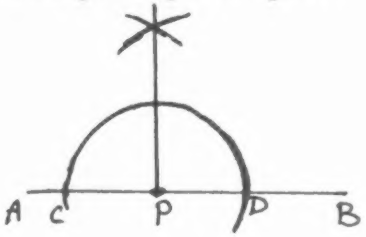
1. Review the following constructions:
    - a) Construct a perpendicular to a given line at a given point on the line (review).
    - b) Construct a perpendicular to a given line from a given point not on the line (review).
    - c) Construct the perpendicular bisector of a segment (review).
    - d) Construct the bisector of a given angle (review).
2. Construct:
    - a) The circumscribed circle of a given triangle.
    - b) The inscribed circle of a given triangle.
    - c) A tangent to a given circle at a given point on the circle.
    - d) A tangent to a given circle from a given point outside the circle.

VOCABULARY: perpendicular, perpendicular bisector, circumscribed circle, inscribed circle

MOTIVATION:

Using a compass and straight edge ruler, construct a line perpendicular to a given line through a given point on the line.

Given: Point P is on line AB.



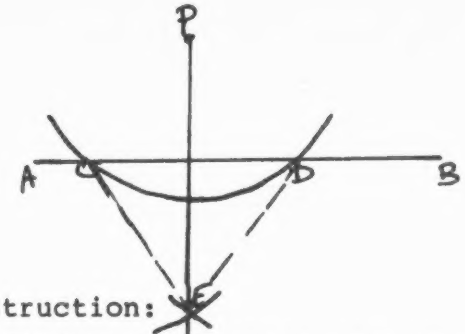
Construction:

1) With P as a center and any convenient radius, construct an arc which intersects $\overline{AB}$ at points C and D.
2) With C and D as centers and with a radius greater than the one used before, construct arcs that intersect at E.
3) Draw $\overleftarrow{EP}$, which is perpendicular to $\overline{AB}$ at P.

DEVELOPMENT:

I. Ask the class to construct a perpendicular to a given line from a given point not on the line (review).

Given: Point P is outside line $\overline{AB}$.

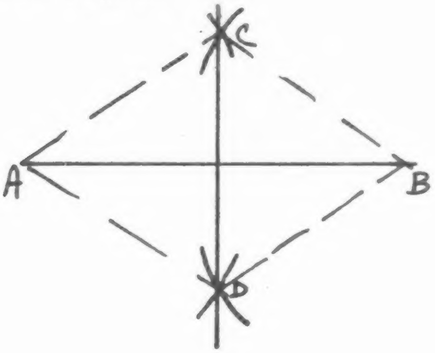


Construction:

1) With P as a center and any radius construct an arc which intersects $\overline{AB}$ in C and D.
2) With C an D as centers and with a radius greater than one-half $\overline{CD}$, construct arcs which intersect in E.
3) Draw $\overline{EP}$, which is perpendicular to $\overline{AB}$.

II. Ask the class to construct the perpendicular bisector of a segment (review).

Given: Line segment AB.



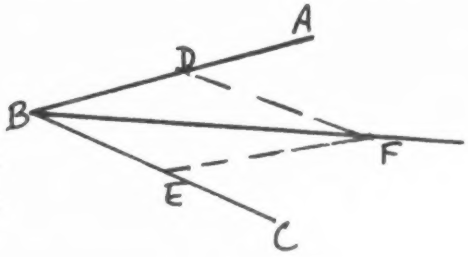
Construction:

1) Open the compass so that the distance is more than half of $\overline{AB}$.
2) Using point A as a center, construct one arc above $\overline{AB}$ and one arc below $\overline{AB}$.
3) Using the same radius and point B as a center, construct another pair of arcs, one of which is above $\overline{AB}$. C and D are points at which the arcs intersect.
4) Draw $\overleftrightarrow{CD}$. The construction makes $\overleftrightarrow{CD}$, the perpendicular bisector of AB, intersect $\overline{AB}$ at E. Therefore $\overline{AE} \cong \overline{EB}$.

III. Ask the class to construct the bisector of a given angle (review).

Given: $\measuredangle ABC$

Construction:



1) With B as a center, construct an arc which intersects $\overline{AB}$ at D and $\overline{BC}$ at E.
2) With D and E as centers and with equal radii of sufficient length, construct arcs that intersect at F.
3) Draw $\overrightarrow{BF}$.
   Since $\measuredangle ABF \cong \measuredangle CBF$, then $\overrightarrow{BF}$ bisects $\measuredangle ABC$.

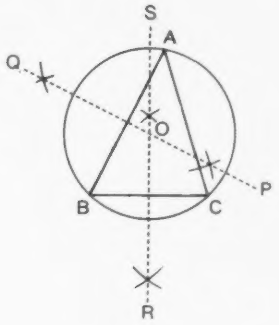
IV. Ask the class to construct the circumscribed circle of a given triangle.

When is a polygon circumscribed about a circle?

Elicit: A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle.

How can we use our previous knowledge to construct this circle? Elicit the construction

Given: $\triangle$ ABC



Elicit: Construction

1) Construct $\overleftrightarrow{PQ}$, the $\perp$ bisector of $\overline{AB}$.
2) Construct $\overleftrightarrow{RS}$, the $\perp$ bisector of $\overline{BC}$.
3) Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at O.
   Since O is on $\overleftrightarrow{PQ}$, $\overline{OA} \cong \overline{OB}$.
   Since O is on $\overleftrightarrow{RS}$, $\overline{OB} \cong \overline{OC}$.
   Therefore $\overline{OA} \cong \overline{OC}$.
   Therefore O is on the perpendicular bisector of $\overline{AC}$.
4) Using $\overline{OA}$ as a radius, draw circle O.
   Triangle ABC is inscribed in ⊙O.

What conclusion can we draw?

The perpendicular bisectors of the sides of a triangle are concurrent, that is, the three perpendicular bisectors meet at one point.

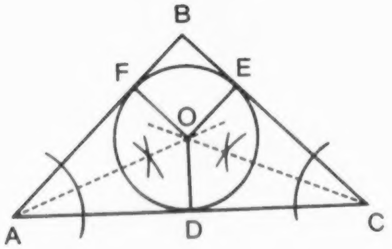
V. Ask the class to construct the inscribed circle of a given triangle:

When is a circle inscribed in a polygon?

Elicit: When a polygon is circumscribed about a circle, the circle is inscribed in the polygon.

Let us now construct the inscribed circle for ABC.

Given: $\triangle ABC$

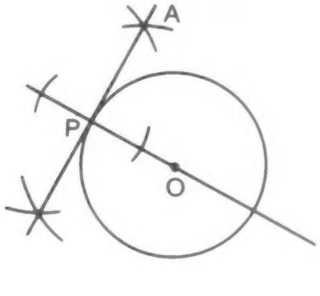


Elicit plan for construction and steps below:

1) Bisect $\angle A$ and bisect $\angle C$. The bisectors of these angles will meet at a point called O.
2) Draw $\overline{OF} \perp \overline{AB}$, $\overline{OE} \perp \overline{BC}$, and $\overline{OD} \perp \overline{CA}$.
3) Since $\angle OFA \cong \angle ODA$ (Right angles are congruent). $\angle FAO \cong \angle DAO$ (An angle bisector divides an angle into 2 congruent angles). $\overline{AO} \cong \overline{AO}$. $\therefore \triangle FAO \cong \triangle DAO$ because aas $\cong$ aas.
4) Similarly, $\triangle DCO \cong \triangle ECO$ because aas $\cong$ aas.
5) $\therefore \overline{OF} \cong \overline{OD}$ Corresponding parts
   $\overline{OD} \cong \overline{OE}$ of $\cong$ triangles are $\cong$.
6) Draw circle O with radii $\overline{OF}$, $\overline{OD}$ and $\overline{OE}$. Since these radii are perpendicular, respectively, to $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$, the sides of the triangle are tangent to circle O and $\triangle ABC$ is circumscribed about circle O. Circle O is also inscribed in the triangle.

VI. Ask the class to construct a tangent to a given circle at a given point on the circle.

Given: Circle O and point P on circle O.

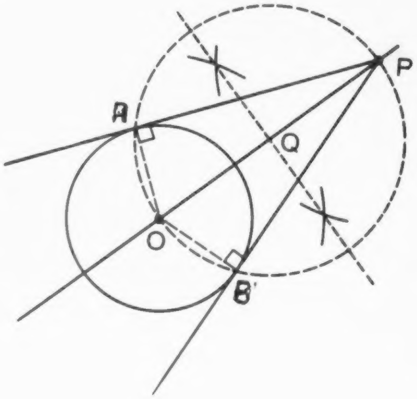


Construction:

1) Draw $\overleftrightarrow{OP}$.
2) Construct a perpendicular to $\overleftrightarrow{OP}$ at P.
3) Since $\overleftrightarrow{AP} \perp \overleftrightarrow{OP}$, line $\overleftrightarrow{AP}$ is tangent to circle O at P.

VII. Ask the class to construct a tangent to a given circle from a given point outside the circle.

Given: Circle O and point P is outside circle O.



Construction:

1) Draw $\overline{OP}$ and bisect it.
2) Q is the midpoint of $\overline{OP}$.
3) Draw a circle with Q as the center and $\overline{OQ}$ as the radius intersecting circle O at A and B.
4) Draw $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$.
5) $\angle OAP$ and $\angle OBP$ are right angles, since $\angle OAP$ and $\angle OBP$ are inscribed in a semicircle of circle Q.
6) $\overleftrightarrow{PA} \perp \overline{OA}$ and $\overleftrightarrow{PB} \perp \overline{OB}$.
   Perpendicular lines meet and form right angles.
7) $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangent to circle O since $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are perpendicular to a radius at its point of contact with the circle.

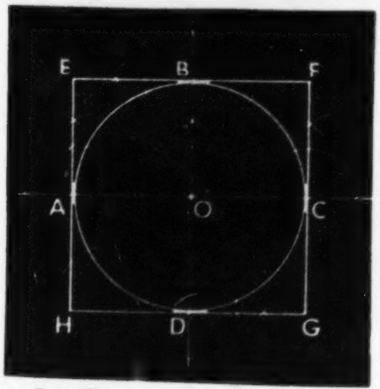
APPLICATIONS:

1. Draw a circle and construct a circumscribed square.
2. Draw a circle and construct an inscribed square.

SOLUTIONS TO APPLICATION PROBLEMS:

1. Draw a circle and construct a circumscribed square.

   Given: Circle O.



Construction:

a) Draw diameter $\overline{AC}$ in circle O.

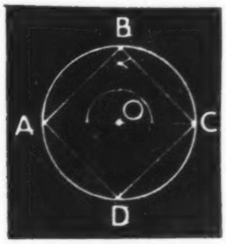
b) At O, construct diameter $\overline{BD}$ perpendicular to diameter $\overline{AC}$.

c) At point A, construct tangent to circle O.
At point B, construct tangent to circle O.
At point C, construct tangent to circle O.
At point D, construct tangent to circle O.
Let tangents intersect at points E, F, G and H.

d) EFGH is a square.

2. Draw a circle and construct an inscribed square.

Given: Circle O.



Construction:

a) Draw diameter $\overline{AC}$ in circle O.

b) At O, construct diameter $\overline{BD}$ perpendicular to diameter $\overline{AC}$.

c) Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

d) ∡AOB = ∡BOC = ∡COD = ∡DOA.
Perpendicular lines meet and form right angles. All right angles are congruent.

e) In a circle, congruent central angles (∡AOB, ∡BOC, ∡COD and ∡DOA) intercept congruent arcs. In a circle, congruent arcs have congruent chords.
Therefore AB = BC = CD = DA and $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA}$.

f) Therefore ABCD is a square.

SUMMARY:

1. Describe how we could locate the center of a given circle by construction.

2. Can a circle be circumscribed about any
   - a) rectangle?
   - b) square?
   - c) parallelogram?
   - d) trapezoid?
   - e) quadrilateral?
   - f) rhombus? Explain your answer.

3. Can a circle be inscribed in any
   - a) rectangle?
   - b) square?
   - c) parallelogram?
   - d) trapezoid?
   - e) quadrilateral?
   - f) rhombus? Explain your answer.

4. Describe how we could construct a tangent to circle O from point X outside of circle O.

LESSON: 60

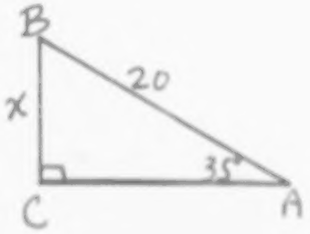
AIM: How do we use trigonometric functions to find sides and angles in a right triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the length of a side of a right triangle when given an acute angle and another side.
2. Find the measure of an acute angle of a right triangle when given the length of two sides of the triangle.
3. Find the trigonometric function of an acute to the nearest ten minutes using the trigonometric tables.
4. State that sine and cosine are cofunctions and use that relationship to find unknown values.
5. Define cotangent of an angle and state that it is the cofunction of tangent.

VOCABULARY: cofunction, cotangent

MOTIVATION:

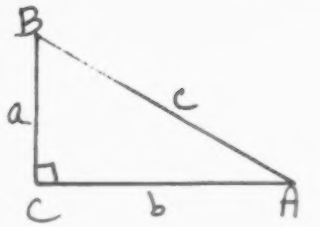


Find the value of X.

In doing this problem, review the definitions of the trigonometric functions and their applications in a right triangle and then state the aim of the lesson.

DEVELOPMENT:

1. Before completing the motivation problem, have the class do the following problem:



Find a) sin A c) cos A
b) sin B d) cos B

From this problem, show that sin A = cos B when A and B are complementary angles.
Define Cofunctions - any trig function of an acute angle is = to the cofunction of its complement.
For example: sin A = cos(90-A)

2. We learned how to find the tangent of an angle and now we define cotangent of an angle - cot A = $\frac{\text{leg adjacent}}{\text{leg opposite}}$
Have students find cot A and cot B and further show that tan A = cot(90-A).
Point out that the prefix, co, means "complement of" when working with trig functions.

3. The class should now peruse the trigonometric tables used in Course III. These tables include the functions of angles to the nearest ten minutes. They also make use of the cofunction relationship in order to find the values for acute angles greater then 45°. Have class look up some values to make sure they understand how to use the tables.

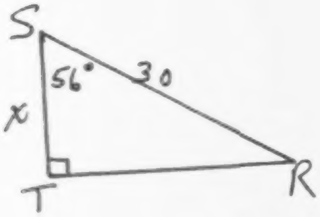
4. Refer back to the motivation problem and complete the solution with the class.

5. Present the following problem: In △ ABC, C=90°, AB=8 and BC=6. Find m<B to nearest ten minutes. Obtain $\measuredangle B = 41°20^1$.

APPLICATIONS:

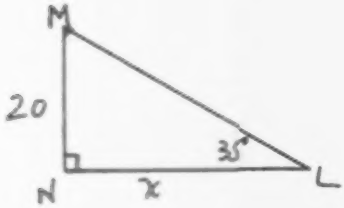
1. Find the value of X: sin(2X+10) = cos 3X
2. Find the value of A: tan(3A-5) = cot(27+A)
3. If tan B = .4320, then find B to the nearest ten minutes.
4. If sin X = .8460, find X to the nearest ten minutes.

5.



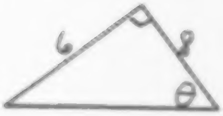
Find value of x
to nearest tenth.

6.



Find x to nearest tenth.

7.



Find θ to nearest 10 minutes.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $2X+10+3X = 90$
$5X = 80$
$X = 16$

2. $3A-5+27+A = 90$
$4A+22 = 90$
$4A = 68$
$A = 17$

3. tan 23°20' = .414
tan 23°30' = .438

$$\begin{array}{r} .4348 \\ \underline{-.4320} \\ .0028 \end{array} \qquad \begin{array}{r} .4320 \\ \underline{-.4314} \\ .0006 \end{array}$$

Since tan B is cosest to tan 23°20', then angle B = 23°20' to the nerest ten minutes.

4. sin X = .8460   sin 64°40' = .8450   sin 64°50' = .8465
x = 64°50'

5. $\cos 56° = \frac{X}{30}$ $\qquad X = 16.776$

$.5592 = \frac{X}{30}$ $\qquad X = 16.8$

6. $\tan 55° = \frac{X}{20}$ $\qquad \cot 35° = \frac{X}{20}$ $\qquad X = 28.562$

$1.4281 = \frac{X}{20}$ $\qquad 1.4821 = X$ $\qquad X = 28.6$

7. $\tan \theta = \frac{6}{8} = .7500$

$\theta = 36°50'$

SUMMARY:

1. What are the definitions of the four trigonometric functions?
2. What does the prefix "co" represent for a trigonometric function?
3. Why are sine and cosine cofunctions?
4. What information about a right triangle must be given in order to find the length of a side?
5. What information about a right triangle must be given in order to find the measure of an acute angle.

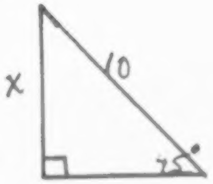
LESSON: 61

AIM: How do we find the trigonometric function values of 30°, 45° and 60° without using a trigonometric table?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the trigonometric function values of 30°, 45° and 60° using right triangles.
2. Evaluate algebraic expressions containing trigonometric functions of 30°, 45° and 60°.
3. Determine that an angle is either 30°, 45°, or 60° when given a triangle with the appropriate side measures.

MOTIVATION:



Find the value of X without using a trigonometric table.

Students may do this problem by either of the two following methods:

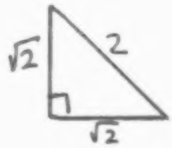
1) Recalling that the leg opposite 45° in a right triangle equals $\frac{1}{2}$ the hypotenuse times the square root of 2.
2) Determining that this is an isosceles right triangle and both legs are equal and then using the Pythagorean Theorem.

Both methods should be discussed.

Point out that it was not necessary to use trigonometric tables in this problem because there is a special relationship between the sides of a right triangle when the given angle is 45°. Further state that there is a special relationship when the given angle is either 30° or 60° and then state the aim of the lesson.

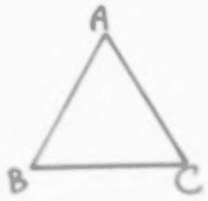
DEVELOPMENT:

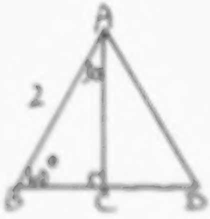
1. Refer back to the motivation problem triangle and ask the class to find the sin 45°, cos 45°, tan 45° and cot 45°. After finding those values ask the class the following question, "What values could we use for the sides of this triangle which would make this problem easier to complete?". Elicit the following triangle:



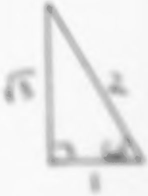
Have the class understand that when working with a 45° angle they are not to use the trigonometric table but must refer to the diagram. Further ask the class, "Which answer is more accurate for sin 45°, $\frac{\sqrt{2}}{2}$ found by using the triangle or .7071 found by using the table?" The class should understand that .7071 is an approximation and not an exact answer.

2. Draw equilateral △ ABC on the board and review the following properties:
   a) All sides are equal in length.
   b) All angles measure 60°.
   Further elicit that an altitude drawn from vertex A will bisect the vertex angle and bisect the opposide side.





Now draw 30°-60°-90° ABC with A=30° B=60° and AB=2. Ask the class to find length of BC. The class should see that BC=1 because if side BC is extended its own length to D and AD is drawn then △ ABD is an equilateral triangle and AC is an altitude. Now have the class find AC in simplest form. They should use the following triangle to find the trigonometric functions of 30° and 60°.



APPLICATIONS:

1. Evaluate sin 45 cos 30 + sin 30° cos 45°
2. Evaluate (tan 30°)^2
3. Evaluate $\sin^2 60° + \cos^2 60°$
4. Evaluate (1 + tan 60°)^2
5. If sin x = $\frac{1}{2}$, then what is the value of x?
6. If tan x = 1, then what is the value of x?
7. Evaluate and express answer with a rational denominator.

   a) $\dfrac{\tan 60°}{1 + \tan 60°}$

   b) $\dfrac{2 \sin 45°}{1 + 2 \cos 45°}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. sin 45 cos 30 + sin 30 cos 45 =

   $\frac{\sqrt{2}}{7} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

   $= \frac{\sqrt{6} + \sqrt{2}}{4}$

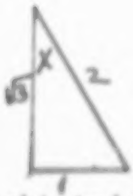
2. $(\tan 30°)^2 = (\frac{1}{\sqrt{3}})^2 = \frac{1}{3}$

3. $\sin^2 60° + \cos^2 60 = (\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = \frac{3}{4} + \frac{1}{4} = 1$

4. $(1 + \tan 60°)^2 = (1 + \sqrt{3})^2 = 1 + \sqrt{3} + \sqrt{3} + 3 = 4 + 2\sqrt{3}$

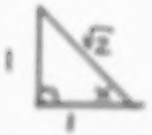
5. sin x = $\frac{1}{2}$ - Draw right triangle — third side is $\sqrt{3}$

   x = 30°



6. tan x = 1 - Draw right triangle — third side is $\sqrt{2}$

   x = 45°



7. Evaluate and express answer with a rational denominator.

a) $$\frac{\tan 60°}{1 + \tan 60°} = \frac{\sqrt{3}}{(1+\sqrt{3})} \cdot \frac{(1-\sqrt{3})}{(1-\sqrt{3})} = \frac{\sqrt{3}-3}{1-3} = \frac{\sqrt{3}-3}{-2} = \frac{3-\sqrt{3}}{2}$$

b) $$\frac{2\sin 45°}{1 + 2\cos 45} = \frac{2\left(\frac{\sqrt{2}}{2}\right)}{1 + 2\left(\frac{\sqrt{2}}{2}\right)} = \frac{\sqrt{2}}{(1+\sqrt{2})} \cdot \frac{(1-\sqrt{2})}{(1-\sqrt{2})} = \frac{\sqrt{2}-2}{1-2} = 2-\sqrt{2}$$

SUMMARY:

1. What triangle should we draw in order to find the trigonometric function values of:
   - a) 30°
   - b) 45°
   - c) 60°

2. a) Find the sin 45° in two different ways?
   b) Which gives a better answer for the sin 45°, the trigonometric table or the triangle? Justify your answer.

LESSON: 62

AIM: How do we draw an angle in standard position?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Draw an angle of any measure in standard position.
2. Identify the quadrant in which an angle belongs.
3. Determine whether angles are coterminal.
4. Identify quadrantal angles.
5. Determine the quadrant of an angle when given a point on the terminal side of the angle.
6. Find the number of degrees in an angle when given the coordinates of a point on the terminal side (limit unknown angles to the quadrantal angles and angles whose reference angles are 30°, 45° and 60°).

VOCABULARY: standard position, coterminal, terminal side, quadrantal angle

MOTIVATION:

1. How many degrees of rotation does the minute hand of a clock go through in a) 20 minutes b) 2 hours?
2. The time on your clock is now 1:55. If you rotate the minute hand 60°, the time on the clock would read the following: a) 2:55 b) 2:15 c) 1:45 d) 2:00

In answering the first question, students will realize that there are instances where rotation may be greater than 360°. In answering the second question students will realize that rotation can be made in 2 directions, clockwise and counter clockwise. In order to develop a full understanding of angles, it is necessary to have a method for representing angles which will account for rotations that are 1) greater than 360° and 2) in either of two directions.

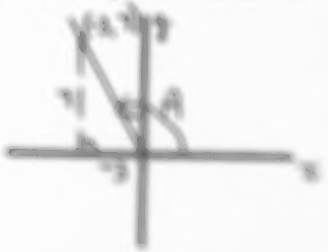
DEVELOPMENT:

1. Review the definition of an angle - The geometric figure formed by 2 rays with a common endpoint. The vertex of the angle is the common endpoint and the 2 rays are called the sides of the angle. We now revise the definition by beginning with the 2 rays in the same position and rotating one of the rays. The fixed ray is called the initial side and the rotated ray is called the terminal side. Counterclockwise rotation is considered positive and a clockwise rotation is considered negative. The standard position of the angle is achieved when the vertex of the angle is placed on the origin and the initial side coincides with the positive part of the x-axis.

2. The coordinate axes divide the plane into 4 areas called quadrants numbered I-IV. An angle is in one of the 4 quadrants. The quadrant of the angle is determined by the quadrant in which the terminal side lies. Have students practice drawing positive and negative angles and determining the quadrant of that angle.

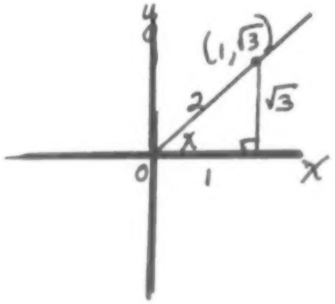
3. In drawing angles in standard positions students should be led to the following results:
   a) Angles of 0°, 90°, 180°, 270° and 360° do not belong to any quadrants and are called quadrantal angles.
   b) Two angles of different measure may have their terminal sides in identical positions. These angles are called coteminal angles.

4. Students should now be asked to determine the quadrant of an angle when given a point on the terminal side. For example - when drawn in standard position the terminal side of angle A contains the point (-3,7). Draw angle A in standard positionand find the quadrant in which angle A belongs. Draw thediagram with the class:



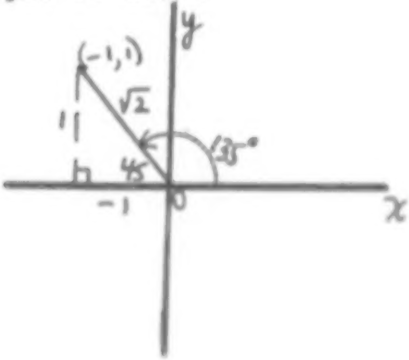
From the point draw a to the x-axis. Show that the length of the vertical leg is equal to the y-coordinate of the point and the horizontal leg is equal to the x-coordinate. State that the legs of the triangle are directed distances so that they may be negative as well as positive.

5. Have the students draw in standard position angle x whose terminal side contains the point $(1, \sqrt{3})$. Ask the class to also find the measure of angle x. The following explanation should be made:

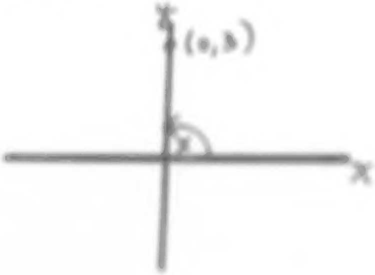


In the diagram the legs of the right triangle are 1 and 3. The hypotenuse is 2. Angle x must be 60°.

Repeat above procedure for (-1, 1). Students should see that the angle in the triangle is 45° but that the given angle is 135°.



Use the point (0, 3).
x = 90°



Develop result that if one of the coordinates of a point on the terminal side is zero, then the angle is a quadrantal angle.

APPLICATIONS:

1. Draw each of the following angles in standard position and state the quadrant of the angle.

   a) 120° b) 260° c) 300° d) 400° e) -110°
   f) -200° g) 500° h) 270°

2. Determine if each pair of angles are coterminal.

   a) 120°, 480° b) -60°, 300° c) 200°, -180°
   d) -90°, 270°

3. In each problem point A is on the terminal side of angle A. Draw angle A in standard position, and find the distance of point A from the origin.

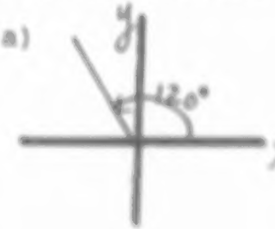
   a) (3, 4) b) (-2, -3) c) (4, -4) d) (0, 3)
   e) ($\sqrt{5}$, 2)

4. In each problem point A is on the terminal side of angle A. Draw angle A in standard position, find the distance of point A from the origin and find the number of degrees in angle A.

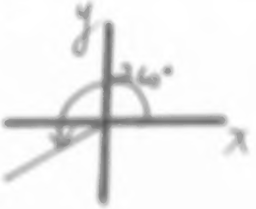
   a) (3, 3) b) ($\sqrt{3}$, 1) c) (4, -4) d) (0, -4)
   e) (2, $-2\sqrt{3}$)

SOLUTIONS TO APPLICATION PROBLEMS:

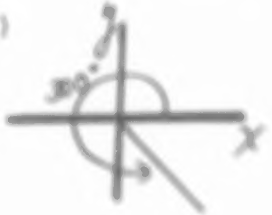
1. a)



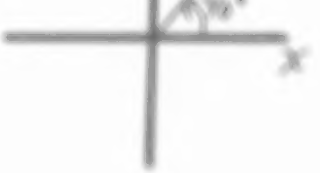
b)



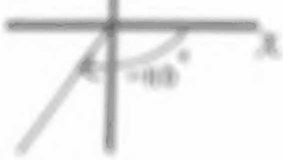
c)



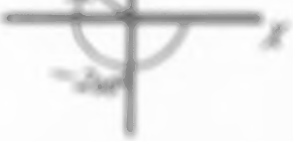
d)



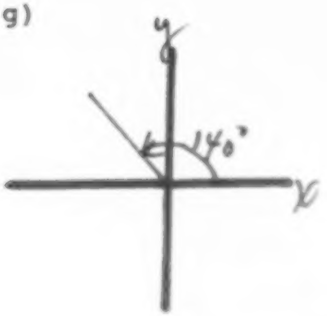
e)



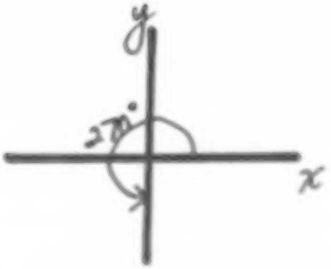
f)



g)



h)



2. a) yes b) yes c) no d) yes

3. a) $r = \sqrt{3^2+4^2} = 5$ b) $r = \sqrt{(-2)^2+(-3)^2} = 13$
c) $r = \sqrt{4^2+4^2} = 4\ 2$ d) $r = \sqrt{0^2+3^2} = 3$
e) $r = \sqrt{(\sqrt{5})^2+2^2} = 3$

4. a) $r = 3\sqrt{2}$; $A = 45°$ b) $r = 2$; $A = 30°$
c) $r = 4\sqrt{2}$; $A = 315°$ or $-45°$ d) $r = 4$; $A = 270°$ or -90
e) $r = 4$; $A = 300°$ or $-60°$

SUMMARY:

1. How do we draw an angle in standard position?
2. When are two angles coterminal?
3. How is a negative angle generated?
4. Which angles do not lie in any quadrant? What are those angles called? What must be true about one of the coordinates of a point on the terminal side of a quadrantal angle?
5. Both coordinates of a point on the terminal side of an angle are negative. In what quadrant does that angle belong?

23. Have the student do Application #5.

APPLICATIONS:

1. Rationalize the denominators:

   a) $\frac{\sqrt{3}}{\sqrt{2}}$ b) $\frac{\sqrt{8}}{3\sqrt{11}}$ c) $\sqrt{\frac{2}{15}}$

2. Find the products and simplify:

   a) $7\sqrt{8} \cdot 3\sqrt{10}$ b) $-5n^2\sqrt{14n} \cdot 2\sqrt{21n^3}$

3. Find the products and simplify:

   a) $3\sqrt{10}(2\sqrt{5}-4\sqrt{2})$ c) $(6-2\sqrt{5})^2$

   b) $(3+ 2)(5-3\sqrt{2})$ d) $(3+2\sqrt{6})(3-2\sqrt{6})$

4. Rationalize the denominators:

   a) $\frac{4}{\sqrt{11}-3}$ b) $\frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}}$

5. Divide and simplify:

   a) $\frac{30\sqrt[3]{128}}{6\sqrt[3]{2}}$ b) $\frac{4\sqrt{108}-2\sqrt{150}}{2\sqrt{6}}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

   b) $\frac{8}{3\sqrt{11}} = \frac{8}{3\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{8\sqrt{11}}{3\cdot 11} = \frac{8\sqrt{11}}{33}$

   c) $\sqrt{\frac{2}{15}} = \frac{\sqrt{2}}{\sqrt{15}} = \frac{\sqrt{2}}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{\sqrt{30}}{15}$

2. a) $7\sqrt{8} \cdot 3\sqrt{10} = 7\cdot 3\sqrt{8\cdot 10} = 21\sqrt{80} = 21\sqrt{16\cdot 5} = 21\cdot 4\sqrt{5}$
   $= 84\sqrt{5}$

   b) $-5n^2\sqrt{14n} \cdot 2\sqrt{21n^3} = -10n^2\sqrt{14n\cdot 21n^3} = -10n^2\sqrt{7\cdot 2\cdot 7\cdot 3n^4}$
   $= -10n^2\cdot 7n^2\sqrt{6}$
   $= -70n^4\sqrt{6}$

3. a) $3\sqrt{10}(2\sqrt{5}-4\sqrt{2}) = 6\sqrt{50}-12\sqrt{20} = 6\sqrt{25\cdot 2}-12\sqrt{4\cdot 5}$
   $= 6\cdot 5\sqrt{2}-12\cdot 2\sqrt{5}$
   $= 30\sqrt{2}-24\sqrt{5}$

b) $$\begin{aligned}(3+\sqrt{2})(5-3\sqrt{2}) &= 3(5-3\sqrt{2}) + \sqrt{2}(5-3\sqrt{2})\\ &= 15-9\sqrt{2} + 5\sqrt{2}-3\cdot 2\\ &= 9-4\sqrt{2}\end{aligned}$$

c) $$\begin{aligned}(6-2\sqrt{5})^2 = (6-2\sqrt{5})(6-2\sqrt{5}) &= 6(6-2\sqrt{5})-2\sqrt{5}(6-2\sqrt{5})\\ &= 36-12\sqrt{5} - 12\sqrt{5} + 4\cdot 5\\ &= 56 - 24\sqrt{5}\end{aligned}$$

d) $$\begin{aligned}(3+2\sqrt{6})(3-2\sqrt{6}) = 3(3-2\sqrt{6})+2\sqrt{6}(3-2\sqrt{6}) &= 9-6\sqrt{6}+6\sqrt{6}-4\cdot 6\\ &= -15\end{aligned}$$

4. a) $$\begin{aligned}\frac{4}{\sqrt{11}-3} = \frac{4}{\sqrt{11}-3}\cdot\frac{\sqrt{11}+3}{\sqrt{11}+3} = \frac{4(\sqrt{11}+3)}{(\sqrt{11})^2-(3)^2} &= \frac{4(\sqrt{11}+3)}{11-9}\\ &= \frac{4(\sqrt{11}+3)}{2}\\ &= 2(\sqrt{11}+3)\\ &= 2\sqrt{11} + 6\end{aligned}$$

b) $$\begin{aligned}\frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{2}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2} &= \frac{\sqrt{12}+2}{6-2}\\ &= \frac{\sqrt{4\cdot 3}+2}{4}\\ &= \frac{2\sqrt{3}+2}{4}\\ &= \frac{2(\sqrt{3}+1)}{4}\\ &= \frac{\sqrt{3}+1}{2}\end{aligned}$$

5. Divide and simplify:

a) $$\frac{30\sqrt[3]{128}}{6\sqrt[3]{2}} = 5\sqrt[3]{64} = 5\cdot 4 = 20$$

b) $$\begin{aligned}\frac{4\sqrt{108}-2\sqrt{150}}{2\sqrt{6}} &= \frac{4\sqrt{108}}{2\sqrt{6}} - \frac{2\sqrt{150}}{2\sqrt{6}}\\ &= 2\sqrt{18} - \sqrt{25}\\ &= 2\sqrt{9\cdot 3} - 5\\ &= 2\cdot 3\sqrt{3} - 5\\ &= 6\sqrt{3} - 5\end{aligned}$$

SUMMARY:

1. What procedure would we use to find the reciprocal of an expression such as $4+2\sqrt{5}$ expressed with a rational denominator?

2. Give an example of two radicals whose product cannot be expressed as a single radical? ie, ($\sqrt[3]{a} \cdot \sqrt{b}$)

* 3. What procedure would you suggest to rationalize the denominator of the fraction $\frac{1}{1+\sqrt[3]{7}}$

LESSON: 21

AIM: How do we find the solution set of a radical equation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write the procedure and apply it to solving radical equations.
2. Check apparent solutions to determine if they satisfy the original equation.

* 3. Solve radical equations which contain two radicals.

VOCABULARY: extraneous roots, radical equations

MOTIVATION:

(A)	(B)
$(x+y)^2$	x^2+y^2

If x and y are real numbers which quantity, A or B, is bigger?

DEVELOPMENT:

1. Ask, "When is quantity A greater than quantity B?"
   Elicit: Illustrate inductively that A>B when x and y are both positive or both negative.
   Example: $(3+4)^2 > 3^2+4^2$
   $49 > 25$
2. Ask, "When is quantity B greater than quantity A?"
   Elicit: Illustrate inductively that B>A when x and y have opposite signs.
   For example: $(-2+5)^2 < (-2)^2+5^2$
   $9 < 29$
3. Ask, "When is quantity A equal to quantity B?"
   Elicit: If either x or y are zero, A=B.
   For example: $(0+-6)^2 = 0^2+(-6)^2$
   $36 = 36$

* 4. Ask: "How can we prove deductively our conjecture?"

Demonstrate: A) $(x+y)^2 = x^2+2xy+y^2$ B) x^2+y^2

Thus we can compare $2xy$ and 0

It follows that the signs of x and y will determine which quantity is larger.

**** Emphasize: $(x+y)^2 \neq x^2+y^2$ ****

5. Ask, "What is the first step in solving the following equations?"

   (a) $3x=6$ (b) $x+6=9$ (c) $\frac{x}{5}=7$

   Elicit: In each case we use inverse operations.

6. Ask, "How would we solve an equation such as:

   (a) $\sqrt{x} = 11$ (b) $\sqrt[3]{2x} = 4$?"

   Elicit: (1) We still use the inverse operation.

   (2) To solve a radical equation with a square root we must square each side of the equation.

   (3) To solve a radical equation with a cube root we must cube each side of the equation.

7. Inform the students: While squaring each side of an equation preserves equality it may introduce apparent solutions which do not satisfy the original equation. For example if $x=5$, then $x^2=25$. The solutions to $x^2=25$ are 5 and -5, but only 5 is a solution to $x=5$. Squaring $x=5$ introduced the extraneous solution $x=-5$.

8. Ask, "How can we determine if an apparent solution is extraneous?"

   Elicit: We must check to see if each apparent solution satisfies the original equation.

9. Model the solution to the radical equation:

   1. Square each side

$$
\begin{aligned}
\sqrt{2x} &= x-4 \\
(\sqrt{2x})^2 &= (x-4)^2 \\
2x &= x^2-8x+16 \\
0 &= x^2-10x+16 \\
0 &= (x-2)(x-8) \\
x-2 = 0 &\text{ or } x-8 = 0 \\
x = 2 & \qquad x = 8
\end{aligned}
$$

Check:

$x=8$: $\sqrt{2\cdot 8} \stackrel{?}{=} 8-4$ $\quad$ $x=2$: $\sqrt{2\cdot 2} \stackrel{?}{=} 2-4$

$\sqrt{16} \stackrel{?}{=} 4$ $\quad$ $\sqrt{4} \stackrel{?}{=} -2$

$4 = 4$ $\quad$ $2 \neq -2$ reject

The solution set is $\{8\}$.

10. Have the student do Application #1.

11. Have the students explore what happens if we square each side of the equation: $\sqrt{2x} + 5 = 3$

    Elicit: Squaring the radical equation in this form does not eliminate the radical: $2x + 10\sqrt{2x} + 25 = 9$

12. Ask, "Is there any other step that we can use that will enable us to eliminate the radical?"

    Elicit: Isolate the radical on one side of the equation.

13. Model the procedure with the equation: $\sqrt{2x} + 5 = 3$

    1. Isolate the $\sqrt{2x}$: $\sqrt{2x} = -2$
    2. Square each side: $(\sqrt{2x})^2 = (-2)^2$

       $2x = 4$

       $x = 2$
    3. Check the apparent solution:

       $x=2$: $\sqrt{2\cdot 2} + 5 \stackrel{?}{=} 3$

       $\sqrt{4} + 5 \stackrel{?}{=} 3$

       $2 + 5 \stackrel{?}{=} 3$

       $7 \neq 3$

    The solution set is $\emptyset$.

14. Ask, "How can we determine that there are no solutions by examining the equation?"

    Elicit: If $y = \sqrt{A}$, then y must be greater than equal to zero. Thus the $\sqrt{2x}$ can never equal -2 for real values of x.

15. Model the solution of: $\sqrt{x+2} + x = 10$

    1. Isolate the $\sqrt{x+2}$: $\sqrt{x+2} = 10-x$
    2. Square each side: $(\sqrt{x+2})^2 = (10-x)^2$

       $x+2 = 100-20x+x^2$

       $0 = x^2-21x+98$

       $0 = (x-14)(x-7)$

       $x-14 = 0$ or $x-7 = 0$

       $x = 14$ $\quad$ $x = 7$

3. Check the apparent solutions: $x = 14$; $x = 7$

$x=14$: $\sqrt{14+2} + 14 \overset{?}{=} 10$ $x=7$: $\sqrt{7+2} + 7 \overset{?}{=} 10$

reject $\sqrt{16} + 14 \overset{?}{=} 10$ $\sqrt{9} + 7 \overset{?}{=} 10$

$4 + 14 \overset{?}{=} 10$ $3 + 7 \overset{?}{=} 10$

$18 \neq 10$ $10 = 10$

The solution set is $\{7\}$.

16. Have the students do Application #2.

17. Ask the students to tell why the second line is incorrect and to write a correct second line of the solution.

(a) $\sqrt{4x} + 6 = 3x$ (b) $\sqrt{2x+2} = 3x-4$

$4x + 36 = 9x^2$ $2x+2 = 9x^2+16$

Elicit: (a) 1. We cannot square term by term.

2. The radical should be isolated: $\sqrt{4x} = 3x-6$

(b) $(3x-4)^2 \neq 9x^2+16$:

Correct $2x+2 = 9x^2-24x+16$

18. Have the students fo Application #3.

* 19. Ask, "How is an equation such as $\sqrt{x-2} + \sqrt{7-x} = 3$ different from those we have solved already?

Elicit: This radical equation has two radical terms.

20. Model the solution: $\sqrt{x-2} + \sqrt{7-x} = 3$

1. Isolate one of the radicals: $\sqrt{x-2} = 3-\sqrt{7-x}$

2. Square each side: $(\sqrt{x-2})^2 = (3-\sqrt{7-x})^2$

$x-2 = 9-6\sqrt{7-x}+(7-x)$

3. Isolate the radical: $2x-18 = -6\sqrt{7-x}$

$x-9 = -3\sqrt{7-x}$

4. Square each side again: $(x-9)^2 = (-3\sqrt{7-x})^2$

$x^2-18x+81 = 9(7-x)$

$x^2-18x+81 = 63-9x$

$x^2-9x+18 = 0$

$(x-3)(x-6) = 0$

$x-3 = 0$ or $x-6 = 0$

$x = 3$ $x = 6$

5. Check the apparent solutions: $x = 3$; $x = 6$

$x=3$: $\sqrt{3-2} + \sqrt{7-3} \stackrel{?}{=} 3$

$\sqrt{1} + \sqrt{4} \stackrel{?}{=} 3$

$1 + 2 \stackrel{?}{=} 3$

$3 = 3$

$x=6$: $\sqrt{6-2} + \sqrt{7-6} \stackrel{?}{=} 3$

$\sqrt{4} + \sqrt{1} \stackrel{?}{=} 3$

$2 + 1 \stackrel{?}{=} 3$

$3 = 3$

The solution set is 3,6

21. Have the students do Application #4.

APPLICATIONS:

1. Find the solution set of $\sqrt{x^2-12} = x-2$.
2. Find the solution set of $\sqrt{2x+1} + 1 = x$
3. Find the solution sets of the following:
   (a) $2x = 3\sqrt{x+3} + 3$ (b) $\sqrt[3]{x-2} + 1 = 4$
4. Have the student find the solution sets of the following.
   (a) $\sqrt{x+2} + \sqrt{3-x} = 3$ (b) $\sqrt{5x-4} - \sqrt{x} = 2$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\sqrt{x^2-12} = x-2$

$(\sqrt{x^2-12})^2 = (x-2)^2$

$x^2-12 = x^2-4x+4$

$4x = 16$

$x = 4$

CHECK $x=4$: $\sqrt{4^2-12} \stackrel{?}{=} 4-2$

$\sqrt{16-12} \stackrel{?}{=} 2$

$\sqrt{4} \stackrel{?}{=} 2$

$2 = 2$

The solution set is $\{4\}$

2. $\sqrt{2x+1} + 1 = x$

$\sqrt{2x+1} = x-1$

$(\sqrt{2x+1})^2 = x^2-2x+1$

$2x+1 = x^2-2x+1$

$0 = x^2-4x$

$0 = x(x-4)$

$x = 0$ or $x-4 = 0$

$x = 4$

CHECK $x=0$: $\sqrt{2\cdot 0+1}+1 \stackrel{?}{=} 0$

reject $\sqrt{1} + 1 \stackrel{?}{=} 0$

$1 + 1 \stackrel{?}{=} 0$

$2 \neq 0$

$x=4$: $\sqrt{2\cdot 4+1} +1 \stackrel{?}{=} 4$

$\sqrt{9} + 1 \stackrel{?}{=} 4$

$3 + 1 \stackrel{?}{=} 4$

$4 = 4$

The solution set is $\{4\}$.

3. a) $2x = 3\sqrt{x+3} + 3$

$2x-3 = 3\sqrt{x+3}$

$(2x-3)^{2} = (3\sqrt{x+3})^{2}$

$4x^{2}-12x+9 = 9(x+3)$

$4x^{2}-12x+9 = 9x+27$

$4x^{2}-21x-18 = 0$

$(4x+3)(x-6) = 0$

$4x+3 = 0$ or $x-6 = 6$

$x = \frac{-3}{4}$ $\quad$ $x = 6$

CHECK $x = \frac{-3}{4}$: $2(\frac{-3}{4}) \overset{?}{=} 3\sqrt{\frac{-3}{4}+3} +3$

reject $\quad$ $\frac{-3}{2} \overset{?}{=} 3\sqrt{\frac{9}{4}} +3$

$\frac{-3}{2} \overset{?}{=} 3(\frac{3}{2}) +3$

$\frac{-3}{2} \overset{?}{=} 3(\frac{3}{2}) +3$

$\frac{-3}{2} \overset{?}{=} \frac{9}{2} +3$

$\frac{-3}{2} \neq \frac{15}{2}$

CHECK $x = 6$: $2(6) \overset{?}{=} 3\sqrt{6+3} +3$

$12 \overset{?}{=} 3\sqrt{9} +3$

$12 \overset{?}{=} 3\cdot 3+3$

$12 = 12$

The solution set is $\{6\}$.

b) $\sqrt[3]{x-2} + 1 = 4$

$\sqrt[3]{x-2} = 3$

$(\ ^{3}\ x-2)^{3} = 3^{3}$

$x - 2 = 27$

$x = 29$

CHECK $x = 29$: $\sqrt[3]{29-2} + 1 \overset{?}{=} 4$

$\sqrt[3]{27} + 1 \overset{?}{=} 4$

$3 + 1 \overset{?}{=} 4$

$4 = 4$

The solution set is 29 .

4. a) $\sqrt{x+2} + \sqrt{3-x} = 3$

$\sqrt{x+2} = 3-\sqrt{3-x}$

$(\sqrt{x+2})^{2} = (3-\sqrt{3-x})^{2}$

$x+2 = 9-6\sqrt{3-x} + (3-x)$

$2x-10 = -6\sqrt{3-x}$

$(2x-10)^{2} = (-6\sqrt{3-x})^{2}$

$4x^{2}-40x+100 = 36(3-x)$

$4x^{2}-40x+100 = 108-36x$

$4x^{2}-4x-8 = 0$

$x^{2}-x-2 = 0$

$(x-2)(x+1) = 0$

$x-2 = 0$ or $x+1 = 0$

$x = 2$ $\quad$ $x = -1$

CHECK $x = 2$: $\sqrt{2+2} + \sqrt{3-2} \overset{?}{=} 3$

$\sqrt{4} + \sqrt{1} \overset{?}{=} 3$

$2 + 1 \overset{?}{=} 3$

$3 = 3$

$x = -1$: $\sqrt{-1+2} + \sqrt{3-(-1)} \overset{?}{=} 3$

$\sqrt{1} + \sqrt{4} \overset{?}{=} 3$

$1 + 2 \overset{?}{=} 3$

$3 = 3$

The solution set is $\{2, -1\}$.

b) $\sqrt{5x-4} - \sqrt{x} = 2$

$$\sqrt{5x-4} = \sqrt{x+2}$$

$$(5x-4)^2 = (\sqrt{x}+2)^2$$

$$5x-4 = x+4\sqrt{x}+4$$

$$4x-8 = 4\sqrt{x}$$

$$2x-4 = 2\sqrt{x}$$

$$(2x-4)^2 = (2\sqrt{x})^2$$

$$4x^2-16x+16 = 4x$$

$$4x^2-20x+16 = 0$$

$$x^2-5x+4 = 0$$

$$(x-1)(x-4) = 0$$

$$x-1 = 0 \text{ or } x-4 = 0$$

$$x = 1 \qquad x = 4$$

CHECK $x = 1$: $\sqrt{5(1)-4} - \sqrt{1} \stackrel{?}{=} 2$

reject $\quad \sqrt{1} - \sqrt{1} \stackrel{?}{=} 2$

$$1 - 1 \stackrel{?}{=} 2$$

$$0 \neq 2$$

CHECK $x = 4$: $\sqrt{5(4)-4} - \sqrt{4} \stackrel{?}{=} 2$

$$\sqrt{16} - \sqrt{4} \stackrel{?}{=} 2$$

$$4 - 2 \stackrel{?}{=} 2$$

$$2 = 2$$

The solution set is $\{4\}$.

SUMMARY:

1. What is the procedure for solving radical equations?
2. Why must we check the apparent solutions to a radical equation?
3. Will there be extraneous solutions to a radical equation with an odd root? Why?

LESSON: 22

AIM: What is an imaginary number?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define "i" and recognize imaginary numbers.
2. Simplify powers of "i".
3. Simplify and combine expressions containing imaginary numbers.

VOCABULARY: pure imaginary number, nested sets, imaginary unit

MOTIVATION:

N = set of natural numbers
W = set of whole numbers
Z = set of integers
Q = set of rational numbers
R = set of real numbers

1. Give an example of an equation which has a solution in W but not in N. (i.e. $x+5=5$)
2. Give an example of an equation which has a solution in Z but not in W. (i.e. $x+5=3$)
3. Give an example of an equation which has a solution in Q but not in Z. (i.e. $3x=5$)
4. Give an example of an equation which has a solution in R but not in Q. (i.e. $x^2-2=0$)

DEVELOPMENT:

1. Ask the students for examples satisfying the conditions of the beginning exercise. From the discussion Elicit: (a) The set are nested that is $N \subset W \subset Z \subset Q \subset R$.
   (b) Mathematicians have developed new number systems to create roots for equations that had none in previously existing systems.
2. Ask, "Can we give an example of an equation that does not have a root in R?"

Elicit: Equations such as $x^2+1=0$ have no real solutions because there is no real number squared which equals -1.

3. Inform the students:
   (1) Early mathematicians considered this equation unsolvable. In the 15th century, mathematicians created a larger system in which the number, -1, exists.
   (2) Rene Descartes (1596-1650) is credited with introducing the term "imaginary" for the square root of any negative number. The term "imaginary" suggests how uncomfortable early mathematician were with these numbers.
   (3) In 1748, Euler is the first to use the symbol "i" to represent the $\sqrt{-1}$, the imaginary unit.
4. Ask: If $i = -1$, what is the value of i^2?
   Elicit: $i^2 = (\sqrt{-1})^2 = \sqrt{-1} \cdot \sqrt{-1} = -1$ since the $(\sqrt{x})^2 = x$.
5. Ask: Find the value of (a) $(4i)^2$ and (b) $(-4i)^2$.
   Elicit: 1. (a) $(4i)^2 = 16i^2 = 16 \cdot -1 = -16$
   (b) $(-4i)^2 = 16i^2 = 16 \cdot -1 = -16$
   2. In general, there are two numbers when squared which will equal a negative number "-a": $i\sqrt{a}$, $-i\sqrt{a}$.
   3. Traditionally $\sqrt{-a} = i\sqrt{a}$
   Examples: $\sqrt{-5} = i\sqrt{5}$, $\sqrt{-13} = i\sqrt{13}$
   $\sqrt{-25} = i\sqrt{25} = 5i$
   $\sqrt{-12} = i\sqrt{12} \quad i\sqrt{4\cdot 3} = i\cdot 2\sqrt{3} = 2i\sqrt{3}$
6. Ask: What is the product of $\sqrt{-6}$ and $\sqrt{-3}$?
   Elicit: (a) The product is not the $\sqrt{18}$. The multiplication rule for radicals does not work when the radicals are both negative.
   Counter example: $\sqrt{-1} \cdot \sqrt{-1} \overset{?}{=} \sqrt{-1 \cdot -1} \overset{?}{=} \sqrt{1} \overset{?}{=} 1$
   However $\sqrt{-1} \cdot \sqrt{-1} = i.i = i^2 = -1$
   thus $\sqrt{-1} \cdot \sqrt{-1} \neq \sqrt{-1 \cdot -1}$
   (b) Change all expressions in the form $\sqrt{-a}$, $a>0$ to $i\sqrt{a}$ before doing any operations.

7. Model the following samples:

a) $\sqrt{-6} \cdot \sqrt{-3} = i\sqrt{6} \cdot i\sqrt{3} = i^2\sqrt{18} = -1\cdot\sqrt{9\cdot 2} = -3\sqrt{2}$

b) $\sqrt{5} \cdot \sqrt{-20} = i\sqrt{5} \cdot i\sqrt{20} = i\sqrt{100} = 10i$

c) $\frac{\sqrt{60}}{\sqrt{-10}} = \frac{\sqrt{60}}{i\sqrt{10}} = \frac{\sqrt{6}}{i} = \frac{\sqrt{6}}{i}\cdot\frac{i}{i} = \frac{i\sqrt{6}}{i^2} = -i\sqrt{6}$

d) $\frac{-\sqrt{-7}}{\sqrt{-14}} = \frac{-i\sqrt{7}}{i\sqrt{14}} = \frac{-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

e) $\sqrt{-16} - 2\sqrt{-4} = i\sqrt{16} - 2i\sqrt{4} = 4i-2i\cdot 2 = 4i-4i = 0$

8. Have the students do Application #1.

9. Inform the students: Numbers in the form bi, b a nonzero real number, are called pure imaginary numbers.

10. Ask: What procedure would you use to simplify i^{1986}?
Elicit: A helpful strategy would be to start with a simpler problem and try to find a pattern.

11. For example: Develop the following table:

$i^1 =$	i	$i^5 = i^4\cdot i = 1\cdot i = i$	$i^9 =$
$i^2 =$	-1	$i^6 = i^4\cdot i^2 = 1\cdot -1 = -1$	etc.
$i^3 = i^2\cdot i = -1\cdot i =$	$-i$	$i^7 = i^4\cdot i^3 = 1\cdot -i = -i$	
$i^4 = i^3\cdot i = -1\cdot i =$	1	$i^8 = (i^4)^2 = 1^2 = 1$	

12. Ask, "Is there a pattern that you have noted?"
Elicit: (a) The values of the powers of i repeat in cycles of 4.
(b) Note, that by using the remainder when dividing by 4. We can determine the value of i^n, "n" a positive integer.
(c) This is a mod_4 system.
(d) This $i^{1986} = (i^4)^{496}\cdot i^2 = 1\cdot -1 = -1$

13. Model the following samples.

a) $i^{61} = (i^4)^{15}\cdot i = i$ b) $i^{103} = (i^4)^{25}\cdot i^3 = -i$

c) $i^{-9} = \frac{1}{i^9} = \frac{1}{(i^4)^2\cdot i} = \frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{i^2} = -i$

14. Have the students do Application #2.

APPLICATIONS:

1. a) $\sqrt{-5}\cdot\sqrt{-80}$ c) $\sqrt{-81}+3\sqrt{-25}+\sqrt{-4}$

b) $\frac{-\sqrt{-48}}{4\sqrt{-3}}$ d) $\frac{\sqrt{75}}{\sqrt{-15}}$

2. a) i^{21} c) $5i^2\cdot 3i^{15}$ e) $i^5\sqrt{-8}$

b) $i^{38}+i^{104}$ d) i^4+i^2 f) $i^3\sqrt{-3}\cdot i^6\sqrt{-12}$

g) i^{-12} h) i^{-7}

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\sqrt{-5}\cdot\sqrt{-80} = i\sqrt{5}\cdot i\sqrt{80} = i^2\sqrt{5\cdot 80} = -1\sqrt{5\cdot 5\cdot 16} = -1\cdot 5\cdot 4$
$= -20$

b) $\frac{-\sqrt{-48}}{4\sqrt{-3}} = \frac{-i\sqrt{48}}{4i\sqrt{3}} = \frac{-\sqrt{16}}{4} = \frac{-4}{4} = -1$

c) $\sqrt{-81}+3\sqrt{-25}+\sqrt{-4} = i\sqrt{81}+3i\sqrt{25}+i\sqrt{4} = 9i+3i\cdot 5+2i$
$= 9i+15i+2i$
$= 26i$

d) $\frac{\sqrt{75}}{\sqrt{-15}} = \frac{\sqrt{75}}{i\sqrt{15}} = \frac{\sqrt{5}}{i} = \frac{\sqrt{5}}{i}\cdot\frac{i}{i} = \frac{i\sqrt{5}}{-1} = -i\sqrt{5}$

2. a) $i^{21} = (i^4)^5\cdot i = 1\cdot i = i$

b) $i^{38}+i^{104} = (i^4)^9\cdot i^2 + (i^4)^{26} = 1\cdot -1+1 = -1+1 = 0$

c) $5i^2\cdot 3i^{15} = 15i^{17} = 15(i^4)^4 i = 15\cdot 1\cdot i = 15i$

d) $i^4+i^2 = 1+-1 = 0$

e) $i^5\sqrt{-8} = i^5\cdot i\sqrt{8} = i^6\sqrt{4\cdot 2} = 2(i^4)^1\cdot i^2\sqrt{2} = -2\sqrt{2}$

f) $i^3\sqrt{-3}\cdot i^6\sqrt{-12} = i^9\cdot i\sqrt{3}\cdot i\sqrt{12} = i^{11}\sqrt{36} = 6\cdot(i^4)^2 i^3 = -6i$

g) $i^{-12} = \frac{1}{i^{12}} = \frac{1}{(i^4)^3} = \frac{1}{1^3} = 1$

h) $i^{-7} = \frac{1}{i^7} = \frac{1}{(i^4)^1\cdot i^3} = \frac{1}{-i} = \frac{1}{-i}\cdot\frac{i}{i} = \frac{i}{-i^2} = i$

SUMMARY:

1. What is a pure imaginary number?
   Why are these numbers called "imaginary"?
2. Prove or disprove:
   a) The set of pure imaginary numbers is closed under multiplication.
   b) The set of pure imaginary numbers is closed under addition.

* 3. Show that the set S = $\{i, 1, -1, -i\}$ is a commutative group.

LESSON: 23

AIM: What are the properties of the set of complex numbers?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and write a complex number in standard a+bi form.
2. State when two complex numbers are equal.
3. Add, subtract, and multiply complex numbers.
4. Identify additive and multiplicative identities and determine additive inverses.

VOCABULARY: complex number, real part, imaginary number

MOTIVATION:

Using the quadratic formula find the roots of:

$$x^2-2x+5 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad a = 1$$

$$= \frac{2\pm\sqrt{4-20}}{20} \qquad b = -2$$

$$= \frac{2\pm\sqrt{-16}}{2} \qquad c = 5$$

$$= 1\pm 2i$$

DEVELOPMENT:

1. Ask, "To which set of numbers do the roots belong?"
   Elicit: The roots are neither real numbers nor are they pure imaginary numbers.
2. Inform the students: (a) The set of pure imaginary numbers together with the set of real numbers form a new set called the complex numbers, denoted by C.
   b) A number that can be written in the form a+bi, where a and b are real numbers is called a complex number.
   c) The number a is called the real part of the complex number, and the number b is called the imaginary part.

Note Teachers should emphasize that b, the coefficient of the imaginary part, is a real number.

d) When the complex number is written in form $a+bi$, it is said to be in standard form.

3. Ask, "Name the real and imaginary parts of these complex numbers."
   - a) $3+2i$
   - b) $-2-7i$
   - c) $4-i\sqrt{6}$
   - d) 5
   - e) $-3i$
   - f) $3+\sqrt{-25}$

4. Ask, "Is a real number a complex number?" If so, when?
   Elicit: (a) A real number is always a complex number because it can be written in the form $a+bi$ where b is equal to zero.
   Examples: $-3 = -3+0i$, $\sqrt{7} = \sqrt{7}+0i$, $\pi = \pi+0i$
   b) If $b \neq 0$, $a+bi$ is called an imaginary number.
   Examples: $5+6i$, $2-i\sqrt{5}$, $7i$

5. Ask, "When is a complex number a pure imaginary number?"
   Elicit: A complex number is a pure imaginary number when $a = 0$ and $b \neq 0$.
   Examples: $0+2i$, $0-i\sqrt{10}$, $-3i$

6. Ask, "When are two complex numbers equal?"
   Elicit: Two complex numbers are equal when the real parts of each are equal and the imaginary parts of each are equal.
   Example: $a+bi = c+di$ if and only if $a = c$ and $b = d$.

7. Have the students do Applications #1 and #2.

8. Ask, "Are the complex numbers closed under addition?"
   Elicit: (a) Before we can answer that we must first define addition in the complex numbers.
   b) If we treat i as a variable and complex numbers as polynomials it follows naturally that:
   $(2+6i)+(3-2i) = (2+3)+(6i-2i) = 5+4i$
   c) Thus $(a+bi)+(c+di) = (a+c)+(b+d)i$ and there is closure under addition.

9. Have the students do Application #3.

10. Ask, "What is the product of two complex numbers?"

Illustrate: (a) $(3+2i)(2-7i) = 6-21i+4i-14i^2$

$= 6-17i-14(-1)$

$= 20-17i$

b) $(a+bi)(c+di) = ac+adi+bci+bdi^2$

$= ac+bd(-1)+adi+bci$

$= (ac-bd)+(ad+bc)i$

11. Have the student do Application #4.

12. Ask, "What are the additive and multiplicative identities in the set of complex numbers?"

Elicit: (a) The additive identity is $0 = 0+0i$ since $(a+bi)+(0+0i) = (0+0i)+(a+bi) = a+bi$.

b) The multiplicative identity is $1 = 1+0i$ since $(a+bi)(1+0i) = (1+0i)(a+bi) = a+bi$.

13. Ask, "What is the additive inverse of a+bi?"

Elicit: The additive inverse is $-a-bi$ since $(a+bi)+(-a-bi) = 0+0i = (-a-bi)+(a+bi)$.

Example: The additive inverse of $3-2i$ is $-3+2i$.

14. Have the students do Applications #5 and #6.

APPLICATIONS:

1. Express $4 - \frac{3}{i}$ in standard form.
2. Solve for x and y:

   a) $x+3i = -7+yi$ b) $(3x-2)+6i = 13+(y-4)i$
3. Find the sum of each pair of complex numbers:

   a) $6+4i$, $3-7i$ c) $-5+2i$, $-5i$

   b) $8+\sqrt{-16}$, $10-\sqrt{-4}$ d) $4\sqrt{2}-3i$, $9\sqrt{2}-i$
4. Express each of the products in standard form.

   a) $(5-2i)(3+4i)$ c) $3i(6-2i)$

   b) $(3-\sqrt{-49})(2+\sqrt{-9})$ d) $(3-2i)^2$
5. Find the product of 7-i and the additive inverse of 3-5i in a+bi form.
6. Show that $(-3+i)$ is a root of $x^2+6x+10 = 0$.
7. Solve for x and y: $i(1-yi) = 4+2xi$.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $4 - \frac{3}{i} = 4 - \frac{3}{i}\cdot\frac{i}{i} = 4 - \frac{3i}{i^2} = 4 - \frac{3i}{-1} = 4+3i$

2. (a) $x+3i = -7+yi$ b) $(3x-2)+6i = 13+(y-4)i$

$x = -7$ $3x-2 = 13$ $6 = y-4$

$y = 3$ $x = 5$ $y = 10$

3. a) $(6+4i)+(3-7i) = 9-3i$

b) $(8+\sqrt{-16})+(10-\sqrt{-4}) = (8+4i)+(10-2i) = 18+2i$

c) $(-5+2i)+(-5i) = -5-3i$

d) $(4\sqrt{2}-3i)+(9\sqrt{2}-i) = 13\sqrt{2}-4i$

4. a) $(5-2i)(3+4i) = 15+20i-6i-8i^2 = 15+14i+8 = 23+14i$

b) $(3-\sqrt{-49})(2+\sqrt{-9}) = (3-7i)(2+3i) = 6+9i-14i-21i^2 =$
$6-5i+21 = 27-5i$

c) $3i(6-2i) = 18i-6i^2 = 18i+6 = 6+18i$

d) $(3-2i)^2 = 9+2(3)(-2i)+4i^2 = 9-12i-4 = 5-12i$

5. Additive inverse of $3-5i$ is $-3+5i$

$(7-i)(-3+5i) = -21+35i+3i-5i^2 = -21+38i+5 = -16+38i$

6. $x^2+6x+10 = 0$; $(-3+i)^2 = 9+2(-3)(i)+i^2 = 9-6i-1 = 8-6i$

$$(-3+i)^2+6(-3+i)+10 \stackrel{?}{=} 0$$
$$(8-6i)+(-18+6i)+10 \stackrel{?}{=} 0$$
$$-10+10 \stackrel{?}{=} 0$$
$$0 = 0$$

7. $i(1-yi) = 4+2xi$

$i-yi^2 = 4+2xi$

$i+y = 4+2xi$

$y+i = 4+2xi$

$y = 4$ $1 = 2x$

$x = \frac{1}{2}$

SUMMARY:

1. What is a complex number?
2. What is the relationship among the sets of pure imaginary, reals, and complex numbers?
3. What are the multiplicative and additive identities in the complex numbers?

* 4. Prove or disprove: The complex numbers form a commutative group under addition.

LESSON: 24

AIM: How do we divide complex numbers?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write the conjugate of a complex number.
2. Find the quotient of two complex numbers and express it in standard form.
3. Write in standard form the multiplicative inverse of a complex number.

* 4. Demonstrate that the complex numbers form a field.

VOCABULARY: complex conjugates, ordered set

MOTIVATION:

Consider the pair of complex numbers 1+i and 1-i. Which is larger, their sum or their product?

DEVELOPMENT:

1. Ask, "Which is the larger of the two quantities?"
   Elicit: Both the sum and the product are equal to 2.
2. Ask, "Can we give an example where the sum and the product of two complex numbers are not equal?
   Elicit: Yes. For example: 3+2i and 5-i. Their sum is 8+i and their product is 17+7i.
3. Ask, "Which is greater, their sum or their product?"
   Elicit: While we have a definition of equality in the complex numbers we do not have a definition of being greater than or less than. We say the complex numbers are not ordered while the real numbers are ordered.
4. Ask, "Which is greater the sum or the product of 3-2i and 3+2i. Elicit: Their sum is 6 and their product is 13. Since these are both real numbers we can state that the product is greater than the sum.

5. Ask, "When will the sum of two complex numbers be a real number?"

   Elicit: If the imaginary parts are additive inverses of each other, the sum will be a real number.

   Example: $(3+2i)+(7-2i) = 10$

6. Ask, "Give examples of two complex numbers whose products will be real numbers?"

   For example: (a) $(2-3i)(2+3i) = 4-9i^2 = 13$

   (b) $(5+4i)(5-4i) = 25-16i^2 = 41$

   (c) $(-2+i\sqrt{3})(-2-i\sqrt{3}) = 4-3i^2 = 7$

   Elicit: The product of complex numbers in the form a+bi and a-bi is a real number. (a^2+b^2)

7. Inform the students that just as $3+\sqrt{2}$ and $3-\sqrt{2}$ were called conjugates, complex numbers in the form a+bi and a-bi are called complex conjugates.

8. Have the students do Application #1.

9. Ask, "What procedure did we use to write the multiplicative inverse of 3+ 2 with a rational denominator?"

   Elicit: It was necessary to multiply the numerator and the denominator by the conjugate of the denominator.

10. Ask, "How could we apply this technique to writing the multiplicative inverse of 3+2i in standard form?"

    Elicit and Illustrate: (a) Write the reciprocal: $\frac{1}{3+2i}$

    (b) Multiply numerator and denominator by the conjugate of the denominator:

    $$\frac{1}{3+2i}\cdot\frac{3-2i}{3-2i}$$

    (c) Simplify and write in the form a+bi: $\frac{3-2i}{9-4i^2} = \frac{3-2i}{13}$

    $$= \frac{3}{13} - \frac{2i}{13}$$

11. Have the students do Application #2.

12. Ask, "How can we write the quotient $\frac{3-i\sqrt{2}}{2+i\sqrt{2}}$ as a complex number in standard form?"

Elicit: Using the same technique we multiply the numerator and denominator by the conjugate of the denominator.

For example:
$$\frac{3-i\sqrt{2}}{2+i\sqrt{2}} = \frac{3-i\sqrt{2}}{2+i\sqrt{2}}\cdot\frac{2-i\sqrt{2}}{2-i\sqrt{2}}$$
$$= \frac{6-3i\sqrt{2}-2i\sqrt{2}-2}{4+2}$$
$$= \frac{4-5i\sqrt{2}}{6}$$
$$= \frac{4}{6} - \frac{5\sqrt{2}}{6}i$$

13. Have the students do Application #3.

* 14. Demonstrate with the youngsters that the complex numbers form a field.

a) Find the multiplicative inverse of a+bi.

b) Show the product of a+bi and its multiplicative inverse is 1, the multiplicative identity.

a) $$\frac{1}{a+bi} = \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

b) $$(a+bi)\cdot\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{a^2}{a^2+b^2} - \frac{ab}{a^2+b^2}i + \frac{ab}{a^2+b^2}i + \frac{b^2}{a^2+b^2}$$
$$= \frac{a^2+b^2+0i}{a^2+b^2}$$
$$= 1$$

APPLICATIONS:

1. Write the conjugates of these complex numbers:

a) $3+4i$ c) $-6i$ e) 9

b) $-6-2i\sqrt{3}$ d) $2i-3$ f) $-\pi$

2. Write the multiplicative inverse of each complex number in standard form:

a) $4-3i$ b) $4i\sqrt{3}-2$ c) $7i$

3. Express each complex number in standard form.

a) $\frac{7-4i}{2-3i}$ b) $\frac{2-4i\sqrt{3}}{3+i\sqrt{3}}$

c) The reciprocal of the conjugate of 5-12i.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $3+4i,\ 3-4i$
   b) $-6-2i\sqrt{3},\ -6+2i\sqrt{3}$
   c) $-6i,\ 6i$
   d) $2i-3,\ -3-2i$
   e) $9,\ 9$
   f) $-\pi,\ -\pi$

2. a) $\frac{1}{4-3i}\cdot\frac{4+3i}{4+3i} = \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25}+\frac{3}{25}i$
   b) $\frac{1}{-2+4i\sqrt{3}}\cdot\frac{-2-4i\sqrt{3}}{-2-4i\sqrt{3}} = \frac{-2-4i\sqrt{3}}{4+48} = \frac{-2-4i\sqrt{3}}{52} = \frac{-1}{26}-\frac{\sqrt{3}}{13}i$
   c) $\frac{1}{7i}\cdot\frac{-7i}{-7i} = \frac{-7i}{+49} = -\frac{i}{7} = 0-\frac{1}{7}i$

3. a) $\frac{7-4i}{2-3i}\cdot\frac{2+3i}{2+3i} = \frac{14+21i-8i+12}{4+9} = \frac{26+13i}{13} = 2+i$
   b) $\frac{2-4i\sqrt{3}}{3+i\sqrt{3}}\cdot\frac{3-i\sqrt{3}}{3-i\sqrt{3}} = \frac{6-2i\sqrt{3}-12i\sqrt{3}-12}{9+3} = \frac{-6-14i\sqrt{3}}{12} = -\frac{1}{2}-\frac{7\sqrt{3}}{6}i$

SUMMARY:

1. What are the steps necessary to write the reciprocal of a complex number in standard form?
2. What are the properties of complex conjugates?
3. Prove or disprove: The product of complex conjugates is less than or equal to their sum.

LESSON: 25

AIM: How can the discriminant help us determine the nature of the roots of a quadratic equation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the value of the discriminant of a quadratic equation.
2. Determine the nature of the roots of a quadratic equation using the discriminant.
3. Determine the number of x-intercepts of the parabola $y = ax^2+bx+c$ using the the discriminant.

VOCABULARY: discriminant, real roots, nature of the roots

MOTIVATION:

Solve the equation $x^2-4x = -13$ for the real solutions.

DEVELOPMENT:

1. Ask, "What is the most efficient way to solve a quadratic equation?"
   Elicit: As a general rule, factoring is the most efficient method. If factoring is not possible, the quadratic formula can be used.
2. Review the use of the quadratic formula by modeling the solution to the motivational exercise.

   $x^2-4x = -13$

   In standard form: $x^2-4x+13 = 0$

   Quadratic formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

   Using a=1, b=-4, c=13 $x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}$

   $x = \frac{4\pm\sqrt{-36}}{2}$
3. Ask, Are there real roots to this equation? Why?
   Elicit: There are no solutions in the set of the real numbers because the $\sqrt{-36}$ is imaginary. To find roots we must consider numbers in the set of complex numbers.

Illustrate: $x = \frac{4\pm\sqrt{-36}}{2}$

$x = \frac{4\pm 6i}{2}$

$x = 2\pm 3i$

4. Have the students do Application #1.

5. Ask, "How can we determine if the roots will be real or imaginary?"
Elicit: (1) If we must take the square root of a negative number the roots will be imaginary thus if b^2-4ac is negative the roots will be imaginary?

6. Inform the students that b^2-4ac is called the **discriminant**.

7. Ask, "What does discriminate mean?"
Elicit: Discriminate means to separate or categorize based upon particular characteristics.

8. Ask, "How can we use the discriminant to determine the "nature of the roots"?"
Elicit: If the discriminant is negative the roots will be imaginary numbers which are complex conjugates. If the discriminant is positive the roots will be real.

9. Ask, "Can we find a number that when added and subtracted from a second number produces equal results?"
Elicit: Yes. The number is zero.

10. Ask, "When will the roots of $ax^2+bx+c = 0$ be equal?"
Elicit: Using the principle above and examining the quadratic formula the discriminant would have to be zero.

11. Organize the preceding information by summarizing:
Let $ax^2+bx+c = 0$ be a quadratic equation with real coefficients.
    a) If $b^2-4ac > 0$, the equation has two real roots.
    b) If $b^2-4ac = 0$, the equation has a real double root.
    c) If $b^2-4ac < 0$, the equation has two imaginary roots which are complex conjugates.

12. Ask, "Given four quadratic equations whose discriminants are (a) -2, (b) 9, (c) 13, (d) -4, how can we determine if the roots are rational or irrational?

Elicit: In order for the roots to be rational the discriminant must be a perfect square of a rational number.

Ans. (a) imaginary (b) rational (c) irrational (d) imaginary

Note While it is necessary for the discriminant to be a perfect square of a rational number, it is not sufficient; a, b, c must be rational numbers as well. For example, the discriminant of $x^2+2\sqrt{7}x+3 = 0$ is 4, but the roots are $-\sqrt{7}\pm 2$.

Thus the roots of $ax^2+bx+c = 0$ are rational if

(1) a, b, c are rational and

(2) b^2-4ac is the square root of a perfect square.

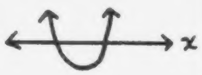
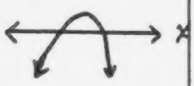
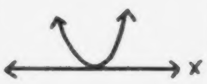
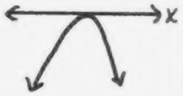
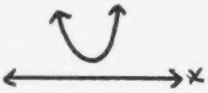
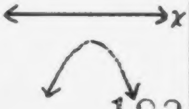
13. Have the students do Applications #2, #3, and #4.

14. Have the students recall that the graphs of the equation $y = ax^2+bx+c$ is a parabola whose axis of symmetry is parallel to the y-axis.
    If $a > 0$, then the graph is concave up.
    If $a < 0$, then the graph is concave down.

15. Ask, "How can we interpret $ax^2+bx+c = 0$ graphically?"
    Elicit: The roots of this equation are the x-intercepts of the parabola.

16. Ask, "How many points of intersection with the x-axis are possible?"
    Elicit: There can be 2, 1, or 0 points of intersection.

17. Have the students draw schematic's for $a>0$ and $a<0$ illustrating the various possibilities.
    Elicit:

Discriminant	a > 0	a < 0	# of X-intercepts
			2
			1
			0

18. Ask, "How does the value of the discriminant relate to the number of x-intercepts?

    Elicit: (a) If $b^2-4ac > 0$, there are two x-intercepts.
    (b) If $b^2-4ac = 0$, there is one x-intercept.
    (c) If $b^2-4ac < 0$, there are zero x-intercepts.

19. Have the students complete the table above.

20. Have the students do Application #5.

APPLICATIONS:

1. Solve the equation $x^2+8 = 4x$ over the set of complex numbers.

2. Write the number and the nature of the roots of the quadratic equation whose discriminant is.
   a) -12 b) 100 c) 13 d) $\frac{1}{4}$ e) $24-4(3)(-2)$
   f) $24-4(3)(2)$ (Assume the coefficients are rational.)

3. Find the value of the discriminant of each equation and write the number and the nature of the roots:
   a) $x^2+4x-4 = 0$ c) $-2x^2+7x = 7$
   b) $3x^2-5x-12 = 0$ d) $8x^2-9x-\frac{1}{2} = 0$

4. Find the values of k for which the equation has one double root? (That is, the roots are equal.)
   a) $x^2+kx+36 = 0$ b) $-2x^2+kx-18 = 0$

5. State whether the graph of the given equation lies entirely above or below the x-axis, intersects the x-axis at 2 distinct points, or intersects the x-axis in 1 point.
   a) $y = x^2+2x+1$ c) $y = -x^2+x+1$
   b) $y = x^2-x+6$ d) $y = -x^2+6x-9$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $x^2+8 = 4x$ $\quad a = 1$

   $x^2-4x+8 = 0$ $\quad b = -4$

   $c = 8$

   $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

   $$x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(8)}}{2(1)}$$

   $$x = \frac{4\pm\sqrt{-16}}{2}$$

   $$x = \frac{4\pm 4i}{2}$$

   $$x = 2\pm 2i$$

2.

a)	-12	2 imaginary roots (complex conjugates)
b)	100	2 real rational roots
c)	13	2 real irrational roots
d)	$\frac{1}{4}$	2 real rational roots
e)	$24-4(3)-2$ $= 48$	2 real irrational roots
f)	$24-4(3)(2)$ $= 0$	1 real double root (rational)

3. a) $x^2+4x-4 = 0$ $\quad a = 1$ $\quad b^2-4ac = 4^2-4(1)(-4)$

   $b = 4$ $\quad = 32$

   $c = -4$

   2 real irrational roots

   b) $3x^2-5x-12 = 0$ $\quad a = 3$ $\quad b^2-4ac = (-5)^2-4(3)(-12)$

   $b = -5$ $\quad = 25+144$

   $c = -12$ $\quad = 169$

   2 real rational roots

   c) $-2x^2+7x = 7$ $\quad a = -2$ $\quad b^2-4ac = 7^2-4(-2)(-7)$

   $-2x^2+7x-7 = 0$ $\quad b = 7$ $\quad = 49-56$

   $c = -7$ $\quad = -7$

   2 imaginary roots (complex conjugates)

   d) $8x^2-9x-\frac{1}{2} = 0$ $\quad a = 8$ $\quad b^2-4ac = (-9)^2-4(8)(-\frac{1}{2})$

   $b = -9$ $\quad = 81+16$

   $c = -\frac{1}{2}$ $\quad = 97$

   2 real irrational roots

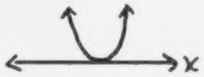
4.

a) $x^2+kx+36 = 0$ $a = 1$
$b^2-4ac = 0$ $b = k$
$k^2-4(1)(36) = 0$ $c = 36$
$k^2-144 = 0$
$(k-12)(k+12) = 0$
$k = 12, k = -12$

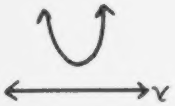
b) $-2x^2+kx-18 = 0$ $a = -2$
$b^2-4ac = 0$ $b = k$
$k^2-4(-2)(-18) = 0$ $c = -18$
$k^2-144 = 0$
$k = \pm 12$

5.

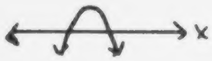
a) $y=x^2+2x+1$ $a=1$ $b^2-4ac=2^2-4(1)(1)$ 1 intersection
concave up $b=2$ $=0$
$c=1$



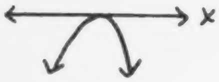
b) $y=x^2-x+6$ $a=1$ $b^2-4ac=(-1)^2-4(1)(6)$ 0 intersections
concave up $b=-1$ $=-23$ entirely above
$c=6$ x-axis



c) $y=-x^2+x+1$ $a=-1$ $b^2-4ac=1^2-4(-1)(1)$ 2 intersections
concave down $b=1$ $=5$
$c=1$



d) $y=-x^2+6x-9$ $a=-1$ $b^2-4ac=6^2-4(-1)(-9)$ 1 intersection
concave down $b=6$ $=36-36$
$c=-9$ $=0$



SUMMARY:

1. How can we use the discriminant to determine the nature of the roots?
2. When will the graph of $y = ax^2+bx+c$ be tangent to the x-axis?

* 3. How can we use the discriminant to determine the number of intersections of $y = ax^2+bx+c$ and $y = k$?

LESSON: 26

AIM: How are the coefficients of a quadratic equation related to the sum and product of its roots?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the sum and product of the roots of a quadratic equation from the coefficients.
2. Write a quadratic equation with integral coefficients given the sum and product of its roots, or the two roots.
3. Find missing coefficients of a quadratic equation given 1 root.

VOCABULARY: integral coefficients, factor theorem

MOTIVATION:

Find a quadratic equation whose roots are 2 and 5.

DEVELOPMENT:

1. Ask: "How is this problem different from the previous exercises we have examined?"

   Elicit: (a) This is the reverse of what we have previously handled. We are asked to write the equation given the roots where formerly we were given the equation and asked to find the roots.

2. Ask: "Can anyone suggest a strategy for solving the beginning exercise?"

   Elicit: Often examining the familiar procedure will suggest to us a procedure for solving this new type of problem.

   For example given the equation: $x^2 + 4x - 21 = 0$

   $$(x + 7)(x - 3) = 0$$

   $$x + 7 = 0 \text{ or } x - 3 = 0$$

   $$x = -7 \qquad x = 3$$

   Therefore the roots of the equation are -7 and 3.

3. Ask: "How can we adapt this procedure to solve our problem?
   Elicit: By reversing the steps we can start with the roots and arrive at a quadratic equation.

4. Model the procedure with the beginning exercise:

(a) Write the roots:	$x = 2$ or $x = 5$
(b) Transform the equations so that one side equals zero:	$x - 2 = 0$ or $x - 5 = 0$
(c) Multiply the factors	$(x - 2)(x - 5) = 0$
(d) Write the equation in standard form	$x^2 - 7x + 10 = 0$

5. Have the students do Application #1.

6. Ask: "Are the coefficients of these equations related to the sum and the products of the roots?

7. Ask: "What strategy would you suggest to find any relationships?"
   Elicit: (1) Organizing the information in a table.
   (2) Examining the table for any patterns.

8. Have the students organize the information into a table.
   For example:

$x^2 + 4x - 21 = 0$; $\{-7, +3\}$	$x^2 - 7x + 10 = 0$; $\{2, 5\}$	$x^2 + 2x - 15 = 0$; $\{-5, 3\}$
SUM = -4 PRODUCT = -21	SUM = 7 PRODUCT = 10	SUM = -2 PRODUCT = -15

9. Ask: What pattern appears to be suggested?"
   Elicit: It appears that the sum of the roots is equal to the opposite of the coefficient of x and the product of the roots is equal to the constant term.

10. Ask: If the equation was $2x^2 - 14x + 20 = 0$, would the pattern still hold?"
    Elicit: This new equation is equivalent to the equation $x^2 - 7x + 10 = 0$ and thus the sum of the roots would still be 7 and the product 10. Thus our conjective will need to be modified!

11. Ask: "How can we transform our conjecture so that the pattern will still hold true?

Elicit: (a) The pattern will still work if the coefficient of x^2 is made to be 1.

(b) The sum of the roots: $r_1 \cdot r_2 = \frac{-b}{a}$

The product of the roots: $r_1 \cdot r_2 = \frac{c}{a}$

12. Ask: "How might we prove these conclusions?"

Proof: Let r_1 and r_2 be the roots of $ax^2 + bx + c = 0$, $a \neq 0$

From the quadratic formula:

$$r_1 = \frac{-b+\sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b-\sqrt{b^2 - 4ac}}{2a}$$

$$r_1 + r_2 = \frac{-b+\sqrt{b^2 - 4ac} \quad + \quad -b-\sqrt{b^2 - 4ac}}{2a}$$

$$r_1 + r_2 = \frac{-2b}{2a}$$

$$r_1 + r_2 = \frac{-b}{a}$$

$$r_1 \cdot r_2 = \frac{-b+\sqrt{b^2 - 4ac}}{2a} \quad \frac{-b-\sqrt{b^2 - 4ac}}{2a}$$

$$r_1 \cdot r_2 = \frac{b^2-(b^2 - 4ac)}{4a^2}$$

$$r_1 \cdot r_2 = \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$r_1 \cdot r_2 = \frac{4ac}{4a^2}$$

$$r_1 \cdot r_2 = \frac{c}{a}$$

13. Ask: "If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, $a \neq o$ what is the equivalent equation in factored form?"

Elicit: Using the factor theorem: $(x - r_1)(x - r_2) = 0$

$$x^2 - r_1x - r_2x + r_1 r_2 = 0$$

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

14) Ask: "How does this compare to the general equation?"

Elicit: (1) By dividing by "a" we obtain $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ and by equating coefficients we see (i) $-(r_1 + r_2) = \frac{b}{a}$ or $r_1 + r_2 = \frac{-b}{a}$

(ii) $r_1 . r_2 = \frac{c}{a}$

15. Have the students do Application #2.

16. Ask: "What procedure could you suggest to write a quadratic equation if we knew the sum and the product of its roots?"

Elicit: The equation would be

x^2 - (sum of the roots)x + product of the roots = 0.

For Example: If the sum of the roots is -6 and the product of the roots is -5, the equation would be: $x^2 + 6x - 5 = 0$

17. Have the students do Application #3.

18. Ask: "How would you show that 6-2i is a root of the quadratic equation $x^2 - 12x + 40 = 0$?"

Elicit: (1) Solve the equation using the quadratic formula.

(2) Substitute to check in the equation.

(3) Use the sum and the product of the roots.

19. Ask "How can we use the sum and the product of the roots?"

Elicit: (1) The other root must be the conjugate of 6-2i because the coefficients are re-l.

(2) $r_1 + r_2 = (6-2i) + (6 + 2i) = 12$

(3) $r_1 . r_2 = (6-2i)(6+2i) = 36-4i^2 = 40$

(4) Thus the equation should be $x^2-12x + 40 = 0$.

20. Have the students do Application #4.

21. Pose: One root of the equation $3x^2 + kx - 8 = 0$ is 4. How can we find the other root and the valve of k?

Demonstrate the solution: Let $r_1 = 4$

$r_1 \cdot r_2 = \frac{c}{a}$ $\qquad$ $r_1 + r_2 = \frac{-b}{a}$

$4 \cdot r_2 = \frac{-8}{3}$ $\qquad$ $4 + (\frac{-2}{3}) = \frac{-k}{3}$

$r_2 = \frac{-2}{3}$ $\qquad$ $\frac{10}{3} = \frac{-k}{3}$

$-10 = k$

22. Have the students do Application #5.

APPLICATIONS:

1. Write a quadratic equation with integral coefficients whose roots are -5 and 3.

2. Find the sum and the product of the roots of the equation:
(a) $2x^2 - 6x + 7 = 0$ (b) $x^2 - 5x + 2 = 2$ (c) $x^2 = 3$

3. Write a quadratic equation in standard form with integral coefficients having the given numbers as its roots.
(a) -2, -9 (c) $\frac{-2}{3}, \frac{1}{2}$ (e) 3-4i, 3+4i
(b) (-6) double root (d) $\sqrt{3}, -\sqrt{3}$ (f) $-8 - 4\sqrt{3}, -8 + 4\sqrt{3}$

4. Show that 3+7i is a root of $x^2 - 6x + 58 = 0$.

5. If r_1 is a root of the equation, find the other root and the value of k.
(a) $2x^2 + kx - 5 = 0$; $r_1 = -1$
(b) $kx^2 - 6x - k = 0$; $r_1 = 2$

SOLUTIONS TO APPLICATIONS PROBLEMS:

1. $x = -5$ or $x = 3$
$x + 5 = 0$ or $x - 3 = 0$
$(x + 5)(x - 3) = 0$
$x^2 + 2x - 15 = 0$

2. (a) $2x^2-6x + 7 = 0$; $a = 2,\ b = -6,\ c = 7$

$r_1 + r_2 = \frac{-b}{a} = -\frac{(-6)}{2} = 3$; $r_1 \cdot r_2 = \frac{c}{a} = \frac{7}{2}$

(b) $x^2 -5x + 2 = 2$

$x^2 -5x = 0 \qquad a = 1,\ b = -5,\ c = 0$

$r_1 + r_2 = \frac{-b}{a} = -\frac{(-5)}{1} = 5$; $r_1 \cdot r_2 = \frac{c}{a} = \frac{0}{1} = 0$

(c) $x^2 = 3$

$x^2 -3 = 0 \qquad a = 1,\ b = 0,\ c = -3$

$r_1 + r_2 = \frac{-b}{a} = \frac{-0}{1} = 0$; $r_1 \cdot r_2 = \frac{c}{a} = \frac{-3}{1} = -3$

3. (a) $r_1 + r_2 = -2 + -9 = -11$; $x^2 + 11x + 18 = 0$

$r_1 \cdot r_2 = (-2)(-9) = 18$

(b) $r_1 + r_2 = -6 + -6 = -12$; $x^2 + 12x + 36 = 0$

$r_1 \cdot r_2 = -6 \cdot -6 = 36$

(c) $r_1 + r_2 = \frac{-2}{3} + \frac{1}{2} = \frac{-1}{6}$; $(x^2 + \frac{1}{6}x - \frac{1}{3} = 0)$

$r_1 \cdot r_2 = \frac{-2}{3} \cdot \frac{1}{2} = \frac{-1}{3}$; $6x^2 + x - 2 = 0$

(d) $r_1 + r_2 = \sqrt{3} + -\sqrt{3} = 0$; $x^2 - 0x - 3 = 0$

$r_1 \cdot r_2 = \sqrt{3} \cdot -\sqrt{3} = -3$; $x^2 - 3 = 0$

(e) $r_1 + r_2 = (3-4i) + (3+4i) = 6$; $x^2 - 6x + 25 = 0$

$r_1 \cdot r_2 = (3-4i)(3+4i) = 9-16i^2 = 25$

(f) $r_1 + r_2 = (-8 -4\sqrt{3}) + (-8 + 4\sqrt{3}) = -16$; $x^2 + 16x + 16 = 0$

$r_1 + r_2 = (-8 -4\sqrt{3})(-8 + 4\sqrt{3}) = 64 - 48 = 16$

4. $x^2 -6x + 58 = 0$

Since the coefficients are real, the other root must be the conjugate of $3 +7i$, $3-7i$.

$r_1 + r_2 = (3 - 7i) + (3+7i) = 6$

$r_1 + r_2 = (3 - 7i)(3 + 7i) = 9 - 49i^2 = 58$

Therefore $r_1 + r_2$ are roots of the equation

$$x^2 -6x + 58 = 0$$

5. (a) $2x^2 + kx - 5 = 0;$

$r_1 = -1$

$r_1 \cdot r_2 = \frac{-5}{2}$

$(-1) \cdot r_2 = \frac{-5}{2}$

$r_2 = \frac{5}{2}$

$r_1 + r_2 = \frac{-k}{2}$

$-1 + \frac{5}{2} = \frac{-k}{2}$

$-2 + 5 = -k$

$3 = -k$

$-3 = k$

(b) $kx^2 - 6x - k = 0;$

$r_1 = 2$

$r_1 \cdot r_2 = \frac{-k}{k} = -1$

$2\, r_2 = -1$

$r_2 = \frac{-1}{2}$

$r_1 + r_2 = \frac{6}{k}$

$2 + \frac{-1}{2} = \frac{6}{k}$

$\frac{3}{2} = \frac{6}{k}$

$3k = 12$

$k = 4$

SUMMARY:

1. How are the coefficients of a quadratic equation related to the sum and the product of the roots?
2. A student solves the equation $2x^2 + 2x - 3 = 0$ and after rounding the solutions to the nearest tenth gets -1.8 and .8. How can she check her solution?

* 3. Is there any relationship between the sum and the product of the roots of a third degree equation and its coefficients?

LESSON: 27

AIM: What is a relation? How do we find the domain and range of a relation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define relation, domain and range of a relation.
2. Find the domain and range of a given relation.
3. Draw a coordinate graph of a given relation.

VOCABULARY: relation, domain, range

MOTIVATION:

The idea of an **ORDERED PAIR** is useful in many situations. Consider the following:

EXAMPLE ONE: We can pair each state with its capital city: (New York, Albany), (Kentucky, Frankfort), (Oregon, Salem) etc. Ask, "Suppose we were given the fictional pair (Duncan, Avon). Which is the capital?"
Note: **This example shows that the ORDER IS IMPORTANT!**

EXAMPLE TWO: Ask, "How are seats usually denoted on theatre tickets?" Elicit that there are 2 parts to the seat designation: 1) The letter giving the correct row 2) the number of the seat in that row. Ask, "Why aren't numbers used for the rows?" Elicit that there would be **CONFUSION** between (3, 9) and (9, 3). **This example too shows that the ORDER IS IMPORTANT!!**

EXAMPLE THREE: When you enter an office building a directory is usually posted **PAIRING OFF** occupants and either the floor or office number. For example: ((Dr. J. Carter, 2), (Dr. G. Ford, 5), (L.B. Johnson Clocks, 3), (Mr. R. Nixon Ghostbuster, 5), (Dr. D. Eisenhower, 2) (Reagan Brothers, 4), (Reagan Brothers, 6), (Truman and Sons Haberdashery, 3), (Hoover Vacuums, 3).)

Here too ORDER counts (but there would be hardly any difficulty if someone reversed the order).

This example illustrates that repetitions can occur in either component! For example, Both Drs. Carter and Eisenhower have a second component of 2.

DEVELOPMENT:

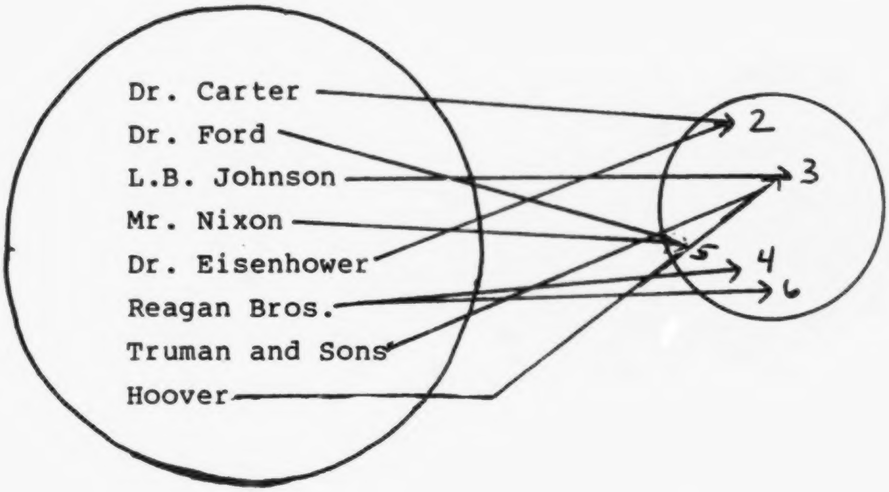
1. Each of the previous three examples is an example of a **"RELATION"**.

   Definition: A **RELATION** is a set of ordered pairs.

2. In order to visualize relations better, there are a few ways to denote a relation **PICTORIALLY**.

   METHOD ONE - AN "ARROW DIAGRAM"

   Taking the relation in example three, we draw:


Dr. Carter
Dr. Ford
L.B. Johnson
Mr. Nixon
Dr. Eisenhower
Reagan Bros.
Truman and Sons
Hoover
2
3
5
4
6

(Note to teacher - When drawing such a diagram on the board, use colored chalk effectively by using the same colors for lines coming together.)

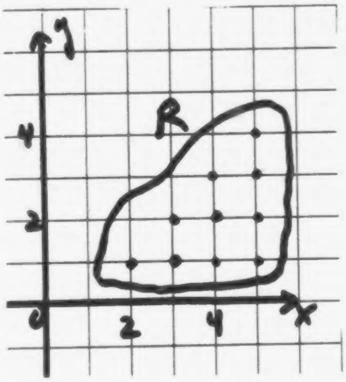
The above diagram leads naturally to the following definitions:

Definitions: 1. In a relation, the set of all the first elements of the ordered pairs is called the **DOMAIN**. The set of all the second elements is called the **RANGE** of the relation.

3. Ask, "Does the relation R = $\{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4)\}$ suggest another pictorial approach for representing a relation?" Elicit:

METHOD TWO - GRAPHING

When the domain AND range consist of real numbers, we can show the relation using a coordinate graph. Here, we see the "picture" or graph of R.



* Note in this relation, the integers 1,2,3,4,5 were used to form ordered pairs. The pairs were created using the **RULE** $x>y$. Often we use the slang "$x>y$ is a relation" when in actuality "$x>y$ is a formula or rule that gives the relation." (Note further the standard terminology uses x for the first element and y for the second element of an ordered pair).

4. Do Applications 1-5.

APPLICATIONS:

1. If R = $\{(3,5), (4,7), (5,9), (0,-1), (-2,-5), (-3,-7)\}$, find
   a) the domain b) the range c) a rule of correspondence between the first and second elements.
2. If R = $\{(-1,2), (0,3), (4,6), (-2,6), (4,5), (2,6)\}$,
   a) find the domain b) find the range c) draw an arrow diagram for the relation. d) draw the graph of the relation.

*3. If the rule is $y=x-5$ and the domain is $\{x: -2\leq x<5\}$,
*a) find the range of the relation **b) redo if the rule is $y=x^2$.

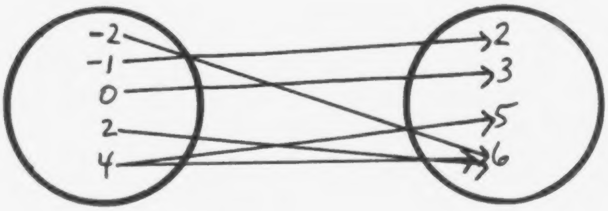
4. If $y=\sqrt{x+2}$, find the largest domain and range so that only real numbers are used.

5. Show that "is the father of" is a "rule" that generates a relation".

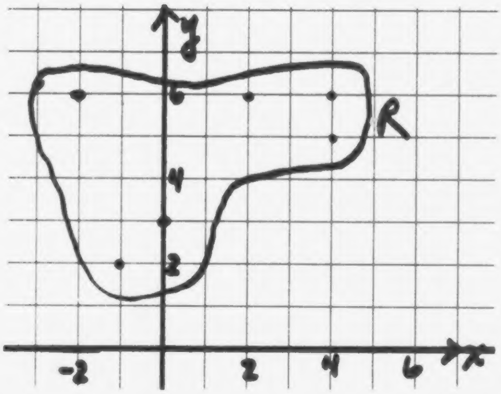
SOLUTIONS TO APPLICATION PROBLEMS:

1. a) Domain $=\{-3,-2,0,3,4,5\}$
   b) Range $=\{-7,-5,-1,5,7,9\}$
   c) $y = 2x-1$

2. a) Domain $=\{-2,-1,0,2,4\}$
   b) Range $=\{2,3,5,6\}$
   c)



d)



3. *a) $\{y: -7\leq y<0\}$ **b) $\{0\leq y<25\}$

*4. Domain $=\{x: x\geq -2\}$ Range $=\{y: y\geq 0\}$

5. Example: Family 1: Father-Philip Children-Harry, Charles, Helen, Blanche, Nettie
Family 2: Father-Morris Children-Stephen, Roslyn

R = {(Philip,Harry), (Philip,Charles), (Philip,Helen), (Philip,Blanche), (Philip,Nettie), (Morris,Stephen), (Morris,Roslyn)}.

R is a relation where the first elements are the fathers and the second elements are the children.

SUMMARY:

1. Ask, "What do we mean by the word RELATION in mathematics?"
2. Ask, "How can we tell the domain and range of a relation by having ONLY the graph?"

LESSON: 28

AIM: What is a function? How does a function differ from a relation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define "function" and distinguish it from a "relation".
2. Determine whether or not a given relation is a function.

VOCABULARY: function

MOTIVATION:

Draw an arrow diagram for each of the following relations. In each case the domain is $\{0,1,4,9,16\}$.

$R1 = \{(x,y): y = x+1\}$

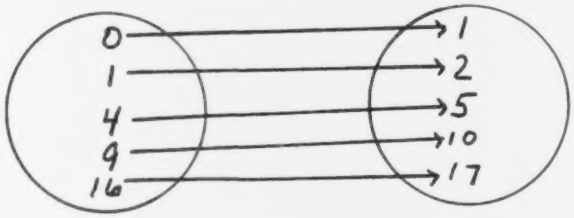
$R2 = \{(x,y): y = 2\}$

$R3 = \{(x,y): x = y^2\}$

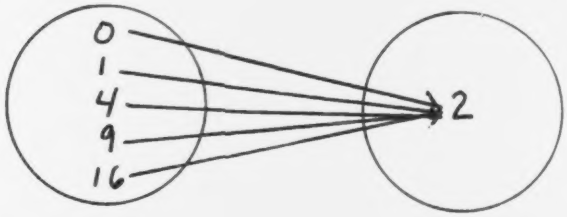
DEVELOPMENT:

1. The above leads to:

R1: $x \rightarrow x+1$

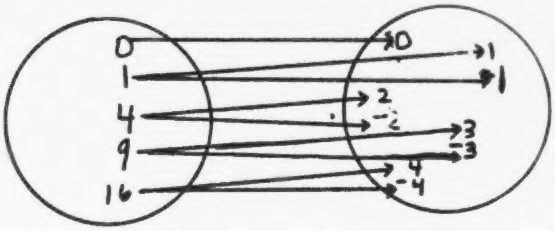


R2: $x \rightarrow 2$



R3: $y^2 = x$ or

R3: $x \rightarrow \pm\sqrt{x}$



2. Show the notation x → x+1 in R1 to mean that the "image" of x is x+1 or y = x+1. Similarly, elicit this alternative notation for R2 and R3.

3. Ask, "What do R1 and R2 have in common?"
   Ask, "How do R1 and R3 differ?"
   Elicit that in R1 and R2 **EACH ELEMENT (x) in the domain IS ASSIGNED EXACTLY ONE element (y) in the range.** This does not happen in R3. Point to 4 and its two assignments 2, -2. In higher mathematics, this property is more desirable! And relations that have this property are called "Functions". (Synonyms - mappings, transformations.)

   Definition: A **FUNCTION** is a relation in which each element in the domain has EXACTLY ONE corresponding value in the range.

4. In R1 and R2 we have relations that ARE functions. Show that the slang "y = x+1 is a function" should really be stated that "y = x+1 DESCRIBES a function" (or "GENERATES").

5. R2 is a special function in which EVERY element of the domain has the same image. We call such a function "a **constant function**".

6. Do Applications 1-4.

7. If need be, do Applications 5 and 6.

8. Do Applications 7-10, 11*.

APPLICATIONS:

1-10: Which of the following are examples of functions?

1. $\{(x,y): y = x\}$
2. $\{(x,y): y = \sin x\}$
3. $\{(x,y): y \geq x\}$
4. $\{(x,y): x = \text{state of U.S.}, y = \text{its capital}\}$
5. $\{(x,y): y = x^2\}$
6. $\{(x,y): y = x^3\}$
7. $\{(x,y): x = \text{wife} \quad y = \text{husband}\}$
   a) Do for legal marriages in the U.S.
   b) Do for legal marriages, - polygamy allowed (1 man permitted several wives).
   c) Do for legal marriages, - polyandry allowed (1 woman permitted several husbands).
8. $\{(x,y): x > 2x\}$
9. $\{(x,y): x = 2y\}$
10. $\{(x,y): x = y^4\}$

* 11. Explain whether or not the following would suffice as a definition of function:
"A function is a set of ordered pairs NO two of which have the same first member."

SOLUTIONS TO APPLICATION PROBLEMS:

1. Function
2. Function (The "sine function").
3. Not a function. Example (6,10) and (6,11) $\in$ relation.
4. Function
5. Function
6. Function

7. a) Function
   b) Function ex. (Raquel, Steve) and (Brooke, Steve) do NOT break the rules for being a function.
   c) Not a function ex. (Raquel, Steve), (Raquel, John) break the rule for being a function!

8. Function

9. Function ($y = \frac{1}{2}x$ or $.5x$)

10. Not a function ex. (16,2) and (16,-2) are in the relation.

11. This is an alternate definition. It does not allow a first element to be assigned to more than one second element.

SUMMARY:

1. Ask, "How does a function differ from a relation?"

2. Ask, "Can a function be a relation?" Why? (It always is.)

3. Ask, "Can a relation be a function?" Why? (It sometimes is.)

LESSON: 29

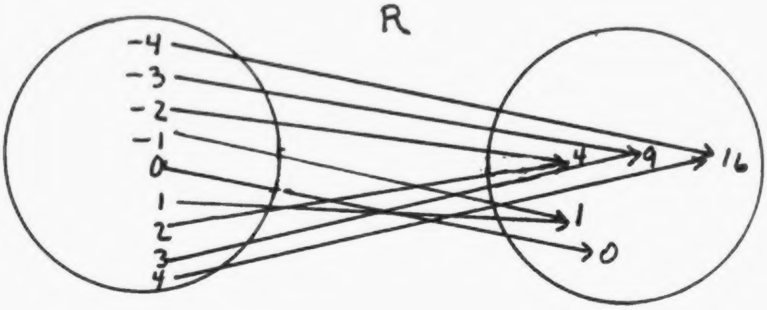
AIM: How do we use functional notation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the y-values for given x-values when a function is given using functional notation.
2. Find the domain of a function when the domain must be restricted.

VOCABULARY: function

MOTIVATION:



Use the above arrow diagram to:

1. List the ordered pairs of the relation R.
2. Determine whether or not R is a function: Explain.
3. Write a rule (formula) that describes the relation.

Elicit:

1. R = $\{(-4,16), (-3,9), (-2,4), (-1,1), (0,0), (1,1), (2,4), (3,9), (4,16)\}$
2. Yes R is a function since each element(x) of the domain corresponds to **EXACTLY ONE** element(y) of the range.
3. $y = x^2$ (Note - we usually denote elements of the domain using "x" and elements of the range using "y".)

DEVELOPMENT:

1. Ask, "Do you remember from yesterday's lesson another notation that means the same thing as $y = x^2$?"
   a) Elicit: $x \to x^2$ or $f\colon\ x \to x^2$.
   b) An advantage of this notation is that ONLY ONE variable(x) appears. We can trace the function(f) and see that "**f maps x to** x^2" or "**x maps to** x^2 under f."

2. Another VERY POPULAR notation is to write $f(x)=x^2$. We say "f or x equals x squared."
   BEWARE: The symbol f(x) DOES NOT mean multiply f times x. It DOES, however, mean that the "y value assigned to x is x^2." NOTE y and f(x) mean the same thing.

3. **MODEL PROBLEM:** If $f(x) = 3x-4$, find:
   a) f(2) b) f(0) c) f(-1)
   Demonstrate: a) $f(2) = 3(2)-4 = 6-4 = 2$ (We substitute 2 for x in the formula.)
   Elicit: b) $f(0) = 0-4 = -4$ c) $f(-1) = 3(-1)-4 = -7$.

4. Do Applications 1 and 2.

5. If $f(x) = \frac{30}{x^2-4}$, find
   a) f(0) b) f(1) c) f(2)
   ($f(0) = -7.5$, $f(1) = -10$)
   Note that f(2) causes a difficulty since substituting 2 for x above leads to zero in the denominator which you know is not allowed. Write on the blackboard.
   When dealing with a given function, if NO domain is given, we accept the domain to be the largest possible subset of real numbers in which each element corresponds to EXACTLY ONE REAL number in the range.
   In the problem above we must exclude 2 from the domain. Ask, "Are there any other number which must be excluded from the domain?" (Yes: -2) Thus the domain is the set of all real numbers excluding 2 and -2.

6. Do Applications 3 and 4*.

7. Find the domain if $y = \sqrt{x}$
   Note that in order for us to obtain REAL numbers in the

range, x CANNOT be negative. Thus the domain is $\{x: x \geq 0\}$ (or "the positive reals and zero").

8. Do Applications 5 and 6 (and 7**).

APPLICATIONS:

1. Given $f(x) = \frac{x^2-3}{3}$, find

   a) $f(3)$ b) $f(0)$ c) $f(-2)$

2. Of $g(x) = |5-2x|$; find

   a) $g(0)$ b) $g(2)$ c) $g(-2)$ d) $g(3)$

3. Find the domain of $f(x)$: $f(x) = \frac{5-x}{x^2-5x+6}$

* 4. Find the domain of $g(x)$: $g(x) = \frac{x-1}{x^2-1}$

5. Find the domain of $h(x)$: $h(x) = \sqrt{x+5}$

6. Find the domain of $f(x)$: $f(x) = \frac{1}{\sqrt{2-x}}$

** 7. Find the domain of $g(x)$: $g(x) = \frac{1}{\sqrt{x^2-9}}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $f(3) = \frac{9-3}{3} = 2$

   b) $f(0) = \frac{0-3}{3} = -1$

   c) $f(-2) = \frac{4-3}{3} = \frac{1}{3}$

2. a) $g(0) = |5| = 5$

   b) $g(2) = |5-4| = 1$

   c) $g(-2) = |5-2(-2)| = 9$

   d) $g(3) = |5-6| = |-1| = 1$

3. $f(x) = \frac{5-x}{(x-3)(x-2)}$ Domain is the set of all reals excluding 2 and 3.

* 4. Note $g(x) = \frac{x-1}{(x-1)(x+1)}$

   Although this reduces to $\frac{1}{x+1}$ we must still exclude 1 from the domain since the original formula is undefined. Note $g(x) = \frac{1}{x+1}$ as long as $x \neq 1$. If $x=1$, $g(x)$ is undefined while $\frac{1}{x+1}$ is defined. This is an important requirement for students who eventually study the Advanced Placement Mathematics Course (both AB and BC). The domain is the set of all reals excluding 1 and -1.

5. $\{x: x \geq -5\}$
6. $\{x: x < 2\}$

**7. $\{x: -3 < x < 3\}$

SUMMARY:

Given the function that "triples" the original number, write this symbolically in as many ways as you can!

Elicit:

1. $y=3x$
2. $f: x \rightarrow 3x$
3. $f(x)=3x$
4. $x \xrightarrow{f} 3x$
5. $f=\{(x,y): y=3x\}$

LESSON: 30

AIM: How do we use the graph of a relation to determine if the relation is a function?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use the vertical line test to determine if a relation is a function.
2. Describe the graph of a constant function.
3. Sketch a graph of the absolute value function.

VOCABULARY: vertical line test

MOTIVATION:

In each case: a) Draw the graph of the relation.
b) Determine whether or not the relation is a function.

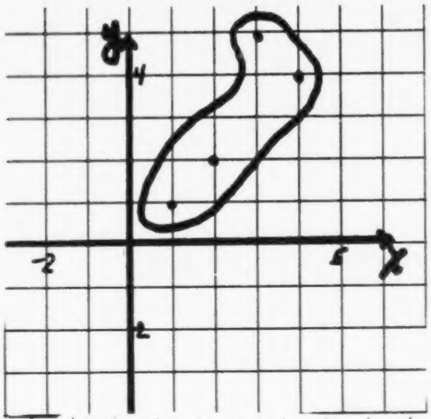
1. $\{(1,1), (2,2), (3,5), (4,4)\}$
2. $\{(x,y): y=x\}$ or more simply, $y=x$
3. $y=10$
4. $x=4$
5. $y=x^2$
6. $y<x^2$
7. $y=|x|$

DEVELOPMENT:

1. Have different students place their graphs on the front board. (In order to speed the lesson up, you can assign different graphs to each row.)

   (Note - If the graph becomes too cluttered, we do not encircle it.)

1)



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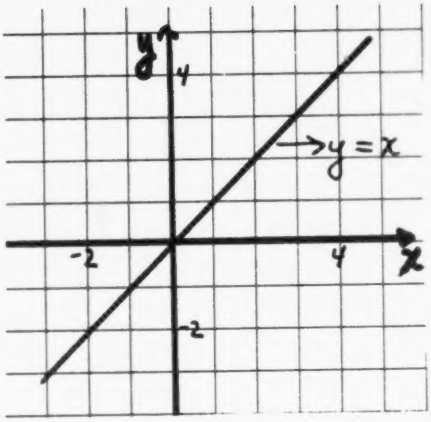

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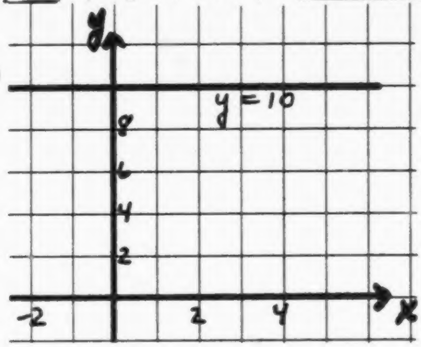
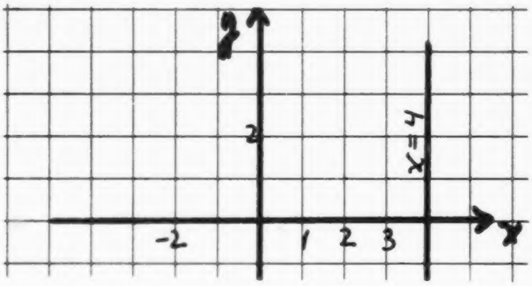


(The graph of a Constant function is a horizontal line)

4)



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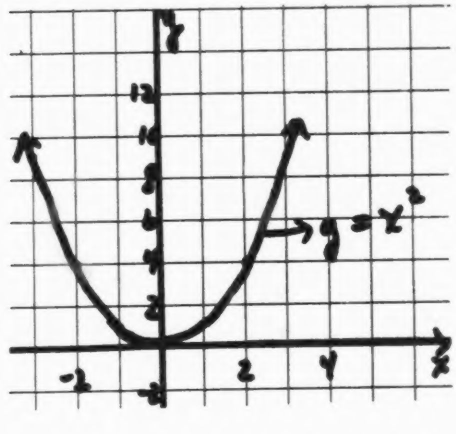

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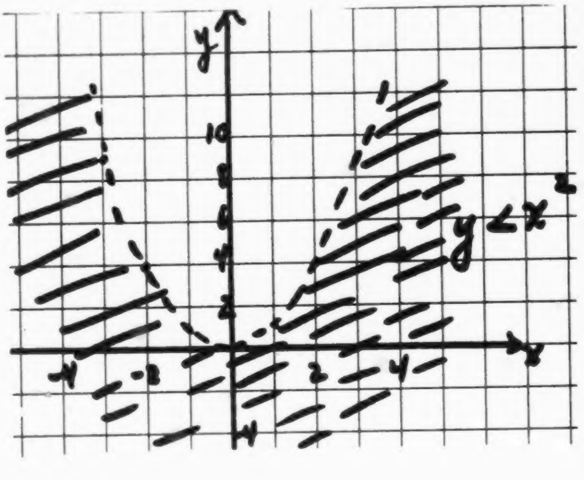

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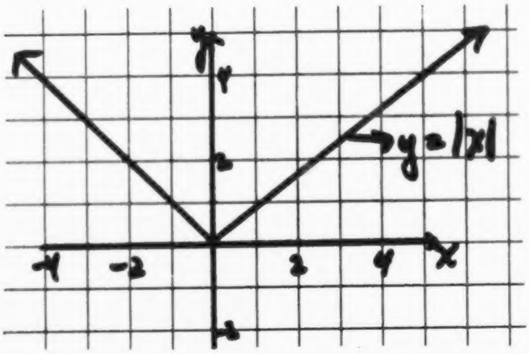


2. Spend time reviewing the graphs especially $y=x^2$ and $y<x^2$. If needed, develop the absolute value graph using a table of values.

3. Elicit that we have functions EXCEPT in 4) $x=4$ and 6) $y<x^2$.

4. Ask, "Can someone come to the board and illustrate why example 6 ($y<x^2$) is NOT the graph of a function?" Hopefully a student will show that two pairs on the graph such as (3,1) and (3,2) violate the requirements for being a function. (3 corresponds to BOTH 1 and 2.)

5. Another way of viewing this is that a vertical line can hit the graph in more than one point. We word this as "the vertical line test".

The Vertical Line Test

If NO vertical line intersects a graph more than once, then the graph is the graph of a function.

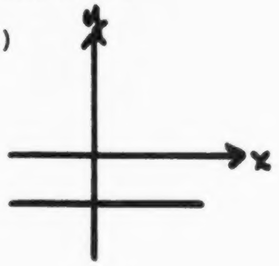
6. Demostrate the Vertical Line Test with examples 1-7 above.

7. Do Applications 1-9.

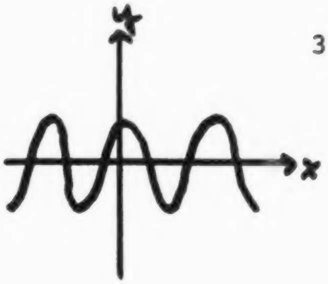
APPLICATIONS:

In each case tell whether or not the graph is the graph of a function:

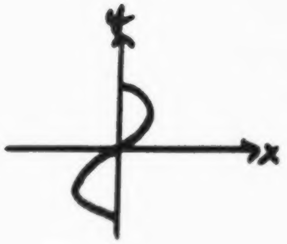
1)



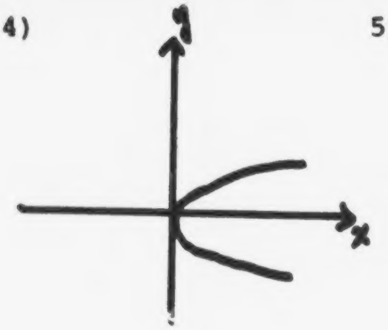
2)



3)



4)



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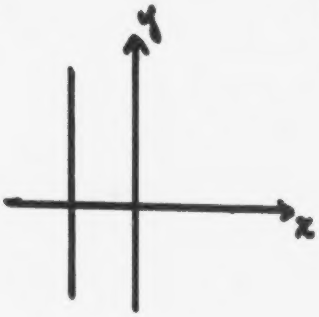

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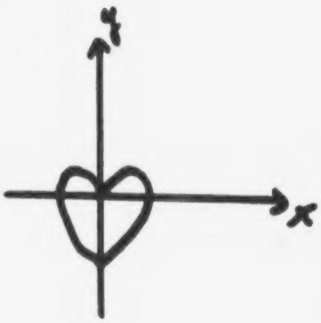


Draw the graphs and determine if the graph is the graph of a function.

7. $x^2+y^2=16$

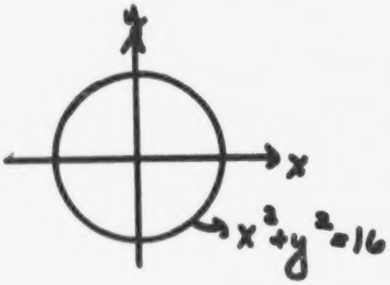
8. $y=-x^2$

9. $x=-y^2$

SOLUTIONS TO APPLICATION PROBLEMS:

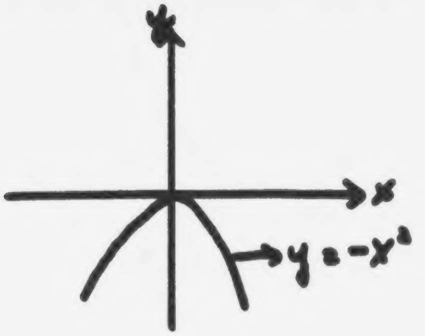
1. Function
2. Function
3. Not a function (fails Vertical Line Test).
4. Not a function (fails Vertical Line Test).
5. Not a function (fails Vertical Line Test).
6. Not a function (fails Vertical Line Test).

7.



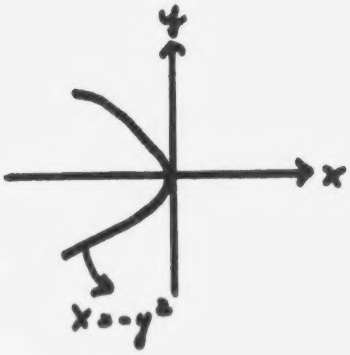
Fails the Vertical Line Test.
Not a function!

8.



This is the graph of a Function!

9.



This fails the Vertical Line Test. Not a function!

SUMMARY:

Ask, "How does the graph of a relation reveal whether or not the relation is a function?"

LESSON: 31

AIM: How do we find the composition of two functions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the result of composing two functions.
2. Handle the two standard notations for compositions of functions.

VOCABULARY: composite, "Composition" of two fractions

MOTIVATION:

Tell the class: This term, one of our neighboring schools got a new chemistry teacher - Mr. Wolfman. Unfortunately he got into trouble last month when they found that he had two mysterious vats containing chemicals. The first vat was for heavy students. After bathing in it, students became HALF the size they had been. The second vat, for thin students, added twenty pounds to whoever bathed in it.

1. Fill in the following chart:

Student's Weight Originally	Vat Used	Student's New Weight
a) 200	first	100
b) 100	second	120
c) 200	second	220
d) 160	first, then second	100
e) 160	second, then first	90

DEVELOPMENT:

1. We can represent this situation algebraically: Add the following entries into the above chart:

f) x	first	2x
g) x	second	x+20

Note that since the result is unique in each case, we are dealing with functions! Let us rewrite f) and g) using functional notation. Elicit:

For Vat 1: $f(x)=\frac{1}{2}x$

For Vat 2: $g(x)=x+20$ (Note we already used "f" so we should use another letter g.)

2. In examples d) and e), BOTH vats were used. Let's examine the situation a little more closely!

   In ex. d) $160 \xrightarrow{f} 80 \xrightarrow{g} 100$ or to summarize
   $160 \longrightarrow 100$

   In ex. e) $160 \xrightarrow{g} 180 \xrightarrow{f} 90$, or to summarize
   $160 \longrightarrow 90$

   These examples illustrate that when one function is performed following another, the two together result in one NEW **function**. We call this new function the **COMPOSITE** of the two functions.

3. There are TWO notations popular in current usage.

   Notation 1 (More Popular): We write $g(f(x))$ or $gf(x)$ to mean **perform f first then perform g**. (We look at the innermost function first.) In the above example, find:

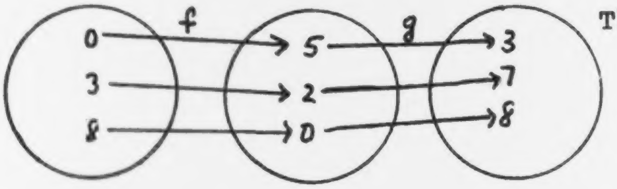
   a) $g(f(200)) = g(100) = 120$ (200 → 100 → 120)
   b) $fg(200) = f(220) = 110$ (200 → 220 → 110)
   c) $gf(x) = g(\frac{1}{2}x) = \frac{1}{2}x+20$
   d) $fg(x) = f(x+20) = \frac{1}{2}x+10$

   Note: THE COMPOSITION OF FUNCTIONS IS NOT COMMUTATIVE!

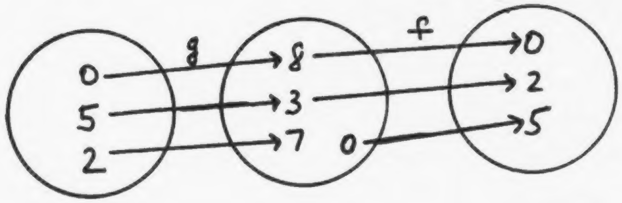
   Notation 2: $(f \circ g)(200) = f(220) = 110$.

   Here we look for the innermost function first (g) and after performing it, we perform the outer function. We can read the above as "f composition g of 200 is 110.

4. Do Application 1 - Use the following arrow diagram's to help:



This shows $g(f(x))$



This shows f(g(x))

5. Do Applications 2 and 3** (honor class only).

6. Tell class that Mr. Wolfman has been fired.

APPLICATIONS:

1. f = {(0,5), (3,2), (8,0)}
   g = {(0,8), (5,3), (2,7)}
   Find a) f(0) b) g(5) c) g(f(0)) d) f(g(0))
   e) (f•g)(5) f) (g•f)(5) g) gf(3)

2. If $f(x) = x^2$ and $g(x) = 2x+1$, find:
   - a) g(f(2))
   - b) f(g(2))
   - c) fg(0)
   - d) g(f(0))
   - e) f(g(x))
   - f) g(f(x))

** 3. We saw that the composition of functions is NOT COMMUTATIVE. Investigate the following group properties for the set of functions:
   - a) Closure
   - b) Associativity
   - c) Identity

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 5 b) 3 c) 3 (show 0 → 5 → 3)
   d) 0 e) 2 f) cannot be done since f(5) is not defined
   g) 7

2. a) $g(f(2)) = g(4) = 9$
   b) $f(g(2)) = f(5) = 25$
   c) $f(g(0)) = f(1) = 1$
   d) $g(f(0)) = g(0) = 1$
   e) $f(g(x)) = f(2x+1) = 4x^2+4x+1$
   f) $g(f(x)) = g(x^2) = 2x^2+1$

** 3. a) The composition of two functions is a function.

b) ex. $f(x) = x^2$
$g(x) = 3x$
$h(x) = x+1$

$$(f(gh))(x) \overset{?}{=} (fg)(h(x))$$

$$f(3x+3) \overset{?}{=} (9x^2)\cdot f(x+1)$$

$$9x^2+18x+9 \overset{?}{=} 9(x+1)^2 = 9x^2+18x+9$$

In general, composition of functions is Associative.

c) Identity Define $I(x) = x$
$fI(x) = f(x)$
$If(x) = f(x)$

These three group properties hold, although after seeing the failure of commutativity, most students may not expect it!

SUMMARY:

If $f(x) = x^2$ and $g(x) = \frac{1}{x}$,

a) Explain the difference in meaning between fg(2) and gf(2).

b) Find both values

Answer: $fg(2)=\frac{1}{4}$, $gf(2)=\frac{1}{4}$

Note: Here we have a rare case where commutativity works; however it is essential that a part a) is explained thoroughly.)

LESSON: 32

AIM: How do we find the inverse of a given relation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the inverse of a given relation or function by roster or formula.
2. Draw the graph of the inverse of a relation by a graph.

** 3. Tell whether the inverse of a relation is a function using "The Horizontal Line Test".

VOCABULARY: inverse relation, inverse function

MOTIVATION:

Tell class: A certain mathematics teacher in our school "Curves" tests that are not just right! He uses two possible functions as test adjustors:

On Very Hard Tests: $f(x)=x+20$
On Very Easy Tests: $g(x)=x-20$

(x is your original test result, f(x) or g(x) is the result recorded in the teacher's marking book.)

Find a) $f(70)$ b) $f(90)$ c) $g(100)$
d) $g(85)$ e) $f(g(90))$ f) $g(f(90))$

(In the last two examples - why should anyone want to perform BOTH functions?)

DEVELOPMENT:

1. Examples (e) and (f) illustrate that f(x) and g(x) here have a special relationship.

   $fg(90)=90$ $gf(90)=90$ or $90 \xrightarrow{g} 70 \xrightarrow{f} 90$; $90 \xrightarrow{f} 110 \xrightarrow{g} 90$

   $fg(20)=20$ $gf(20)=20$ or $20 \xrightarrow{g} 0 \xrightarrow{f} 20$; $20 \xrightarrow{f} 40 \overset{g}{\text{---}} 20$

   Elicit that $f(g(x)) = x$ and $g(f(x)) = x$

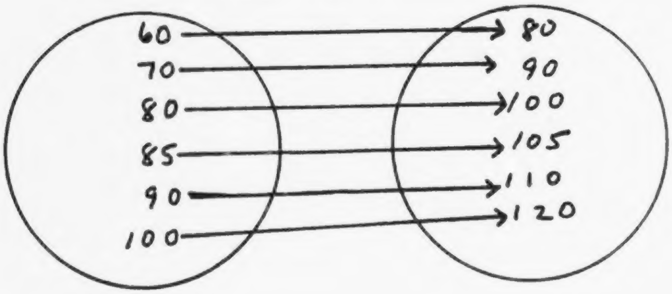
2. This means that the two functions performed successively have "NO effect" on the number you started with (One function "UNDOES" the other).

** Note: The "Identity Function", $I(x)=x$ or $(y=x)$. Thus $f(g(x)) = g(f(x)) = I(x) = x$. (Optional **)

3. Let us clarify this special case further.
Suppose the two sets of grades we obtained were:
"Lower" set of grades = $\{60,70,80,80,85,90,100\}$
"Higher" set of grades = $\{80,90,100,100,105,110,120\}$
We can show the two functions with arrow diagrams and sets as follows:

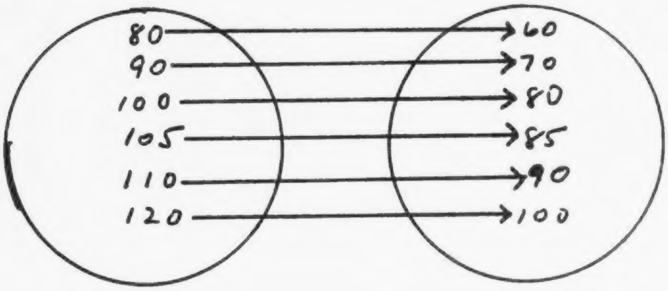
$f: x \rightarrow x+20$

$f=\{(60,80),(70,90),(80,100),(85,105),(90,110),(100,120)\}$



$g: x \rightarrow x-20$

$g=\{(80,60),(90,70),(100,80),(105,85),(110,90),(120,100)\}$



Ask, "How do the ordered pairs of f and g compare?"
Elicit: "They are the reverses of one another."

4. The above illustrates two new ideas which we define:
Definitions: 1. If the ordered pairs of the relation g are the "REVERSE" of the ordered pairs of the relation f, then the two relations are said to be INVERSES of one another.

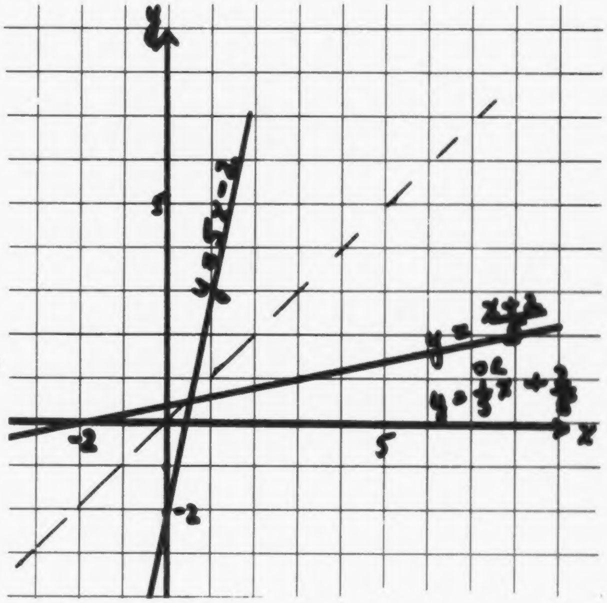
2. If $f(x)$ and $g(x)$ are two FUNCTIONS such that $gf(x)=fg(x)=x$ then f and g are said to be INVERSE FUNCTIONS. We write $g(x)=f^{-1}(x)$. (Note that $f^{-1}(x)$ DOES NOT MEAN $\frac{1}{f(x)}$.)

5. **PROBLEM:** Find the inverse of $y = 5x-2$.
Solution: In the inverse, the ordered pairs are the reverse of the ordered pairs of the original. In other words, "the x-coordinate and y-coordinate of each pair are interchanged". Thus we can write:
ORIGINAL: $y=5x-2$ INVERSE: $x=5y-2$ (or solving for y, we get $y = \frac{x+2}{5}$) Note here both are functions!

6. Graph the original and inverse function in example 5.
Note that: A function and its inverse are always reflections of each other about the line $y=x$.

7. Do Applications 1, 2, 4, 7.

8. With good classes if time permits, do Applications 3, 5, and 6.



APPLICATIONS:

1. Find the inverse of (a) $\{(1,2), (2,4), (3,4)\}$
(b) $y=2x-5$ (c) $y=x^2$

2. Which of the above a) are functions?
b) Have inverses that are functions?

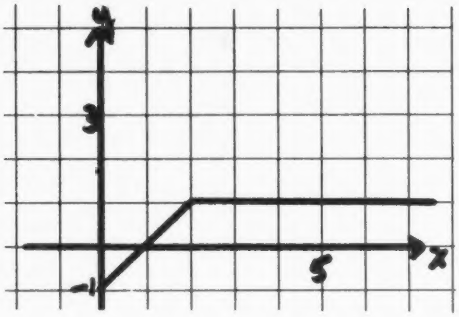
* 3. What restriction should we make on $y=x^2$ so that its inverse is also a function.

4. Draw the graphs in 1(c) and 2(c).

** 5. Ask, "What graphical test did we use to see if the original relation was a function?" Can you invent a test we can use on a graph to see if its inverse is also a function?

** 6. Make up two functions that are inverses of one another. Draw the arrow diagram for $f^{-1}f(x)$. Ask, "What special property(ies) does your arrow diagram illustrate"?

7. Draw the graph of the inverse of this relation:



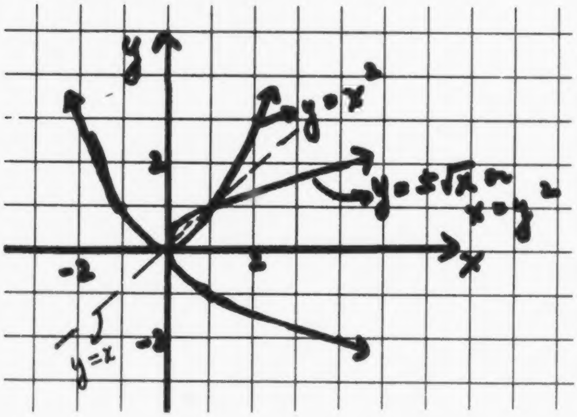
SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\{(2,1), (4,2), (4,3)\}$
   b) $y = \frac{1}{2}x+\frac{5}{2}$
   c) $x=y^2$ or $y=\pm\sqrt{x}$

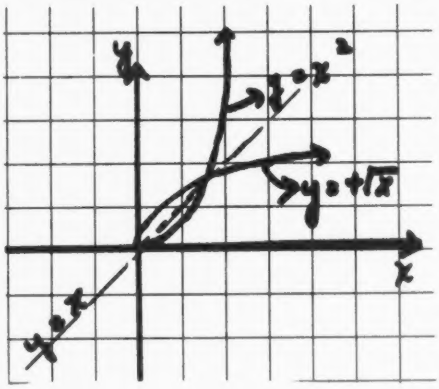
2. All three of the original relations are functions. Only the inverse of (b) is a function.

* 3. If we restrict the domain of $y=x^2$ to $\{x:x\geq 0\}$ then we have a function whose inverse is a function. Graphically, we can see this more clearly:

No Restrictions



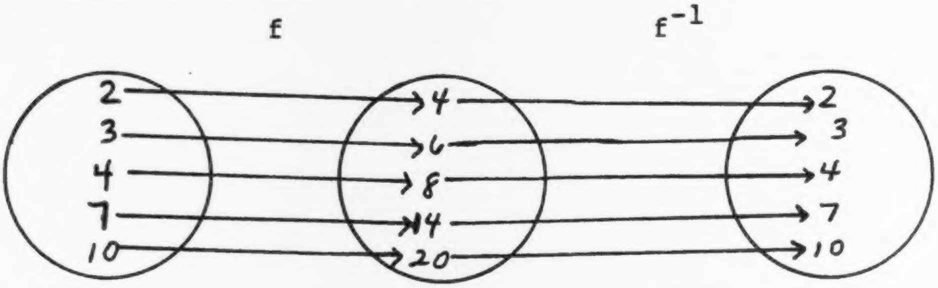
Original Domain = $\{x:x\geq 0\}$



4. See "No restrictions" above in 3.

** 5. Horizontal Line Test: If NO horizontal line can intersect the graph of a relation MORE than once, then its inverse IS a function.

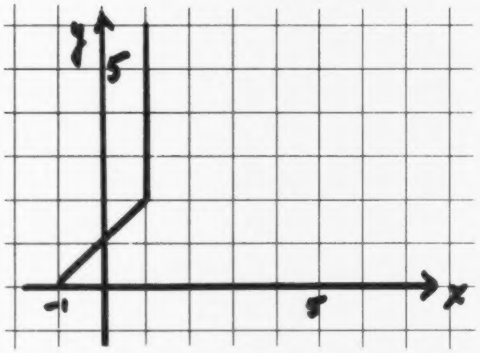
** 6. Example of an answer you can accept:



In f: Every y must have exactly ONE pre-image in its domain in order for f to have an inverse-function. Such a function is said to be "one-one" pronounced "one-to-one".

(Demonstrate with $y=x^2$ using the $\{\text{Reals}\}$ as domain. This idea shows why this function has no inverse function unless the original domain is restricted.)

7.



SUMMARY:

Ask: 1) "What does it mean to say that two functions f and g are inverses of one another?"

2) "Give an example of a function whose inverse is NOT a function."

LESSON: 33

AIM: How do we use the method of linear interpolation to approximate the value of a function?

PERFORMANCE OBJECTIVES: The student will be able to...

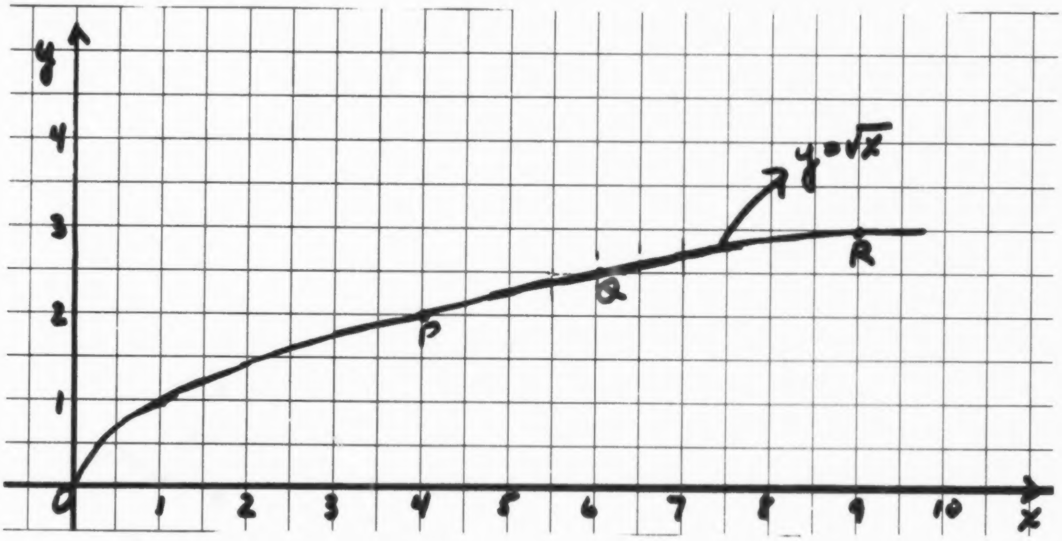
Approximate functional values on a curve by approximating the curve with a line using the method of linear interpolation.

VOCABULARY: linear interpolation

MOTIVATION:

1. Draw the graph of $y=\sqrt{x}$.
2. Use your graph to approximate $\sqrt{6}$.

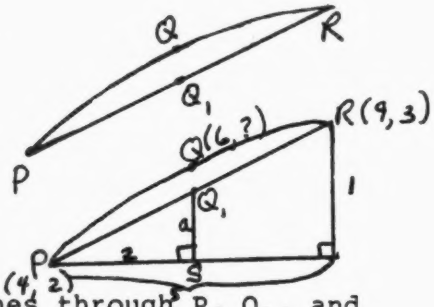
DEVELOPMENT:



1. If the class has forgotten, remind them that the graph of the equation $x=y^2$ was a parabola - the top half being the graph of $y=+\sqrt{x}$ and the bottom half - the graph of $y=-\sqrt{x}$.
2. We can approximate $\sqrt{6}$ using a good graph: 2.5 (see above). Tell the class: For many functions you will use in this course, you will NOT have a good graph readily available. Neither will you have a calculator!

Note that $\sqrt{4} < \sqrt{6} < \sqrt{9}$ or $2 < \sqrt{6} < 3$
On the graph we are dealing with three points P(4,2), Q(6,?) and R(9,3). If you ignore the rest of the graph, these three points are a) fairly close to one another and b) almost on a straight line. When this occurs we can use the "METHOD OF LINEAR INTERPOLATION".

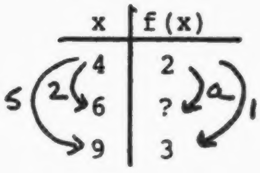
3. The basis of this method is that since P, Q and R are almost on a straight line, we can approximate the curve by the straight line connecting P and R (see diagram). We will find the y-value at Q_1 (on the line) to approximate the y-value at Q (on the curve).



4. We draw horizontal and vertical lines through P, Q_1, and R as shown. Ask, "What is the relationship between ΔPSQ_1 and ΔPTR?" Fill in the following lengths: PS(2), PT(5), Q_1S(a) and RT(1). Ask, "What proportion can we use as a result of having similar triangles?" Elicit one such proportion as $\frac{a}{1} = \frac{2}{5}$ and get a = .4. Ask, "What are the coordinates of Q_1?" Elicit that to get the y-coordinate, we add 2 + .4 and therefore Q_1(6,2.4). Note Q and Q_1 have the same x-coordinates. Since Q_1 is approximating Q our answer is 2.4.

5. Ask, "Do you think we MUST go through all these steps each time?" No - "We just did it to explain the basis behind the method of linear interpolation. (Note "linear" since we estimated the curve by a line, "interpolation" since we used two "outer" points to estimate the coordinates of an "inner" point (vs extrapolation).)

We can EASILY set up the proper proportion using a table:



x	f(x)
4	2
6	?
9	3

REMEMBER: After finding "a" we must add it to the first value of f(x) to get our approximation.

$$\frac{2}{5} = \frac{a}{1}$$

6. Do Applications 1, 2, 3.

APPLICATIONS:

1. Use the method of linear interpolation to approximate:
   1. $\sqrt[3]{25}$ to the nearest tenth.
   2. Sin 12°13' to four decimal places. (Be sure students have trigonometric tables.)
   3. Cos 31°22' to four decimal places.

SOLUTIONS TO APPLICATION PROBLEMS:

1.

		x	$f(x) = \sqrt[3]{x}$		
		8	2	a	
19	17	25	2+a		1
		27	3		

$\frac{17}{19} = \frac{a}{1}$

So a ≈ .895 or .9

f(25) ≈ 2.9

2.

		x	f(x)=sin x		
		12°10'	.2108	a	
10	3	12°13'	.2108+a		28
		12°20'	.2136		

$\frac{3}{10} = \frac{a}{28}$

So a = .84 ≈ 8.0

f(12°13') ≈ .2116

Note: As long as we're careful, the decimal can be eliminated. (The 28 really represents 28 ten-thousandths.)

3.

		x	f(x)=cos x		
		31°20'	.8542	a	
10	2'	31°22'	.8542-a		-16
		31°30'	.8526		

$\frac{2}{10} = \frac{a}{-16}$

a = -3.2 ≈ -3

Thus Cos 31°22' ≈ .8539

Note: The cosine function decreases here. Hence the change was represented as -16. BEWARE!! The change a must be added to the first functional value. Sometimes a can be negative as we see here.

SUMMARY:

Ask, "When can we use the method of Linear Interpolation?"

LESSON: 34

AIM: What is a transformation?

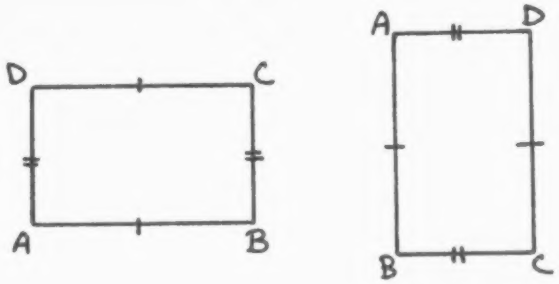
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and illustrate a transformation.
2. Define and illustrate the image of a point.
3. Define and illustrate a line reflection and line symmetry.
4. Express the notation for a line reflection.
5. Identify letters and shapes that have line symmetry.

VOCABULARY: transformation of the plane, reflection in a line, line symmetry

MOTIVATION:

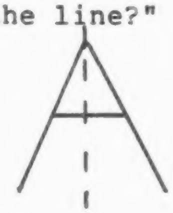
Ask the students to explain what is similar and what is different between figures I and II.



Elicit: The figures are congruent but their positions are different.

DEVELOPMENT:

1. Using the motivation problem establish that what is different between the two figures is that the position of the figure has changed or been moved. Describe this as a transformation.

2. Define a transformation of the plane as a one-to-one correspondence between the points in a plane that demonstrates a change in position or a fixed position for each point in the plane. Demonstrate several other transformations.
3. Using a handout or an overhead projector, establish a point A and a line k. Ask, "If we folded the paper (transparency) along the line k, where would point A lie?" Elicit: On the other side of line k (call the point A'). Establish that A' is the reflection of point A in line k. This point, A', is called the image of point A. Repeat similar exercise with $\triangle$DEF.
4. Do Application #1.
5. Introduce notation $r_k(A) = A'$. (The reflection of point A in line k is A'.)
6. Define a reflection in a line as a transformation of the plane such that
    (a) If point P is not on line k, the image of P is P', where line k is the $\perp$ bisector of PP'.
    (b) If point P is on k, the image of P is P.
7. Illustrate the construction of line reflection from part 6. on the board or overhead projector.
8. Ask, "Given the letter A, can we draw a line which passes through the letter which allows a reflection of every point on one side of the line to have its image fall onto points on the other side of the line?"
Elicit: Yes. (Demonstrate ).
9. Review the axis of symmetry as applied to the parabola from Sequential Mathematics Course II. Define line symmetry as a result of a line making a figure its own image under a reflection in a line. Illustrate vertical and horizontal symmetry.

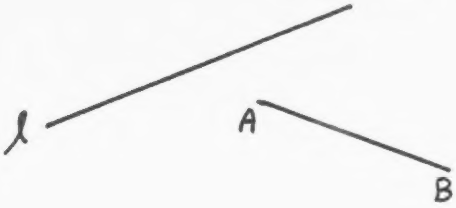
10. Do Application #2.

11. Do Application #3.

12. Do Application #4.

APPLICATIONS:

1. On a piece of paper copy $\overline{AB}$ and line ℓ. Then sketch the image of $\overline{AB}$ under a reflection in line ℓ.

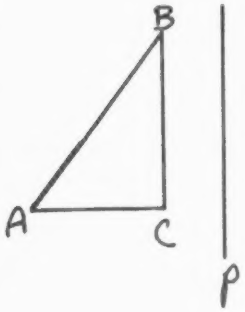


2. a) Find other letters which have line symmetry.
   b) Which of the following words has line symmetry?
   1) RADAR 2) TOOT 3) MOM 4) DEED 5) DAD

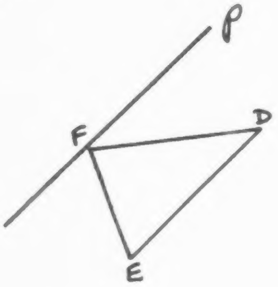
3. Draw three shapes which have line symmetry.

4. Copy the given figure and line p. Sketch the image of the given figure under a reflection in line p. Label the transformation.

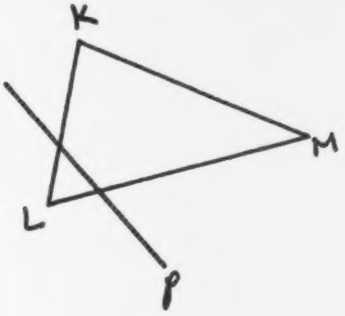
a)



b)

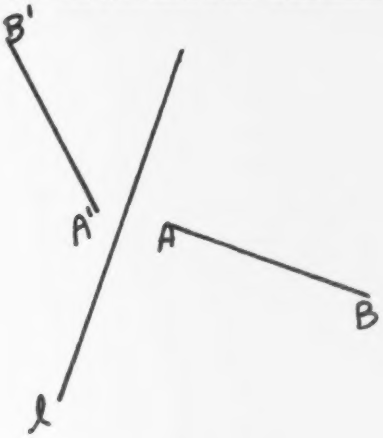


c)



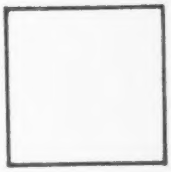
SOLUTIONS TO APPLICATION PROBLEMS:

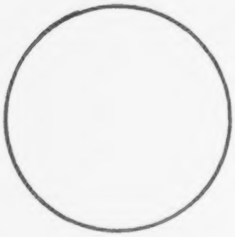
1.

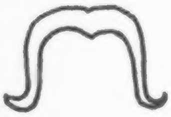


2. a) E, X, H, O, T b) TOOT, MOM, DEED

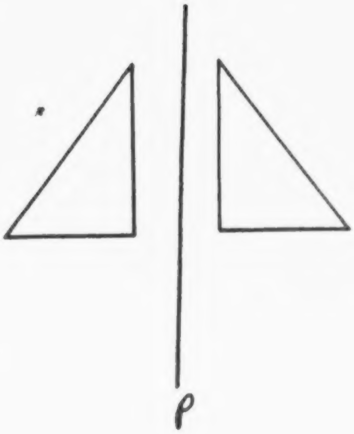
3.



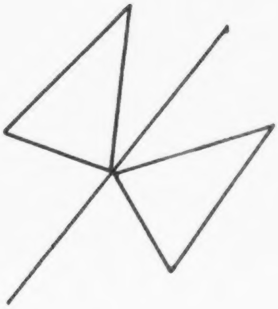




4. a)



b)



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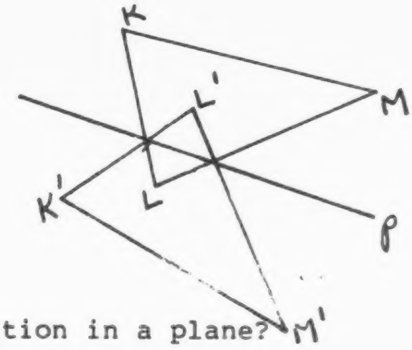


SUMMARY:

1. What is a transformation in a plane?
2. What is a reflection in a line?
3. What is meant by line symmetry?

LESSON: 35

AIM: How can we use coordinate geometry to show properties of line reflections?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Plot geometric figures in the coordinate plane given coordinates of their vertices.
2. Find the image of a given set of points after reflection in any of the following lines:
   x = any integer, y = any integer, $y = x$, $y = -x$.
3. Demonstrate that distance, angle measure, parallelism, collinearity and midpoint are properties preserved under a line reflection.
4. Define and illustrate orientation.

VOCABULARY: orientation, isometry

MOTIVATION:

Graph triangle ABC if the coordinates of its vertices are A(1,6), B(4,3) and C(2,-1). Reflect triangle ABC over the y-axis.

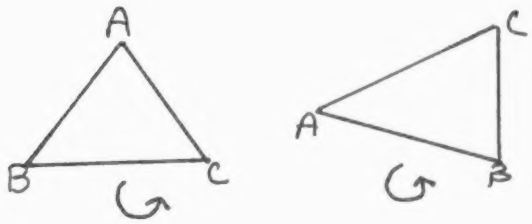
DEVELOPMENT:

1. Using the motivation problem ask: "What are the coordinates of A', B' and C' the image of triangle ABC?" Elicit: A'(-1,6), B'(-4,3) and C'(-2,-1).
   Ask, "What rule can we follow that will represent the image of coordinate (x,y) under a reflection in the y-axis?" Elicit: The x-coordinate is opposite and the y-coordinate remains the same.
2. Establish the notation for a reflection in the y-axis:
   $r_{y\text{-axis}}(x,y) = (-x,y)$
3. Ask, "What can be said about the relative sizes of the triangles?" Elicit: They are congruent. Establish then that one property preserved under a line reflection is

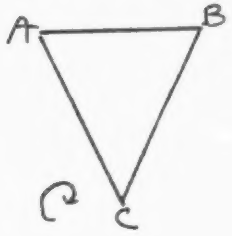
distance. Use the distance formula to show this property.

4. Ask, "What other properties are preserved under a line reflection?" Elicit: Angle measure, collinearity, midpoint, and parallelism. Use slope, midpoint and distance formulas to show these properties are true.
5. Ask, "What are the coordinates of the vertices of quadrilateral DEFG under a reflection in the x-axis if the coordinates are D(1,2), E(4,-5), F(3,0), G(0,-3)?" Elicit: D'(1,-2), E'(4,5), F'(3,0), G(0,3). Ask, What rule does this suggest for reflecting a point over the x-axis?" Elicit: $r_{x\text{-axis}}(x,y) = (x,-y)$.
6. Do Application #1.
7. Ask, "What rule would you suggest for reflecting a point over the line y = x?" Elicit: $r_{y = x}(x,y) = (y,x)$.
8. Do Application #2.
9. Define orientation as preserving the order of the vertices after a transformation. Illustrate orientation with the following diagrams.

Preserves Orientation



Does Not



Define isometry as a transformation that preserves distance. Show that a line reflection does not preserve orientation.

10. Do Application #3.
11. Refer back to part 7. Ask, "What is the relationship between the images and the original points?" Elicit: The coordinates of the images interchange the x and y

values of the coordinates of the original points. Establish that the points represent the inverse of the original points. Refer to the function $F(x) = x + 3$. When we write the equation of its inverse, we interchange the x and y values and get $f^{-1}(x) = x - 3$. Note: If we refected the graph of the line $y = x + 3$ over the line $y = x$, we would have the graph of the equation $y = x - 3$. (The inverse of the original line.)

APPLICATIONS:

1. Reflect triangle ABC over the line $y = 4$. If the coordinates of the vertices are A(-1,2), B(-4,1) and C(-3,-2), what are the coordinates of the images of points A, B, and C? What rule can we suggest for $r_{y=4}(x,y)$?

2. Reflect triangle DEF over the line $y = -x$. If the coordinates of the vertices are D(1,2), E(4,1) and F(3,-2), what are the coordinates of the images of points D, E and F? What rule can we suggest for $r_{y=-x}(x,y)$?

3. Graph quadrilateral ABCD if its vertices are A(2,1), B(5,-2), C(9,4) and D(6,7). Graph A'B'C'D' if $r_{x\text{-axis}}(ABCD)$. Determine whether the properties of distance, parallelism, midpoint and orientation are preserved.

SOLUTIONS TO APPLICATION PROBLEMS:

1. A'(-1,6), B'(-4,7), C'(-3,10).
   $r_{y=4}(x,y) = (x,8-y)$

2. D'(-2,-1), E'(-1,-4), F'(2,-3)
   $r_{y=-x}(x,y) = (-y,-x)$

3. A'(2,-1), B'(5,2), C'(9,-4), D'(6,-7)
   Using slope, parallelism is preserved.
   Using the distance formula, distance is preserved.
   Using the midpoint formula, midpoint is preserved.
   By observation, orientation is not preserved.

SUMMARY:

1. What is meant by orientation?
2. What properties are preserved under a line reflection?
3. What properties are not preserved under a line reflection?

LESSON: 36

AIM: What is meant by point reflection and point symmetry?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and illustrate point reflection.
2. Express the notation for point reflection.
3. Define and illustrate point symmetry.
4. Identify letters and shapes that have point symmetry.

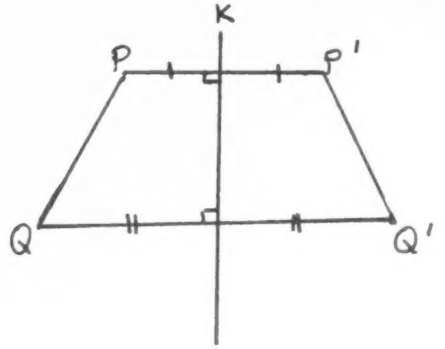
VOCABULARY: point reflection, point symmetry

MOTIVATION:

Write the definition of line symmetry and give an example.

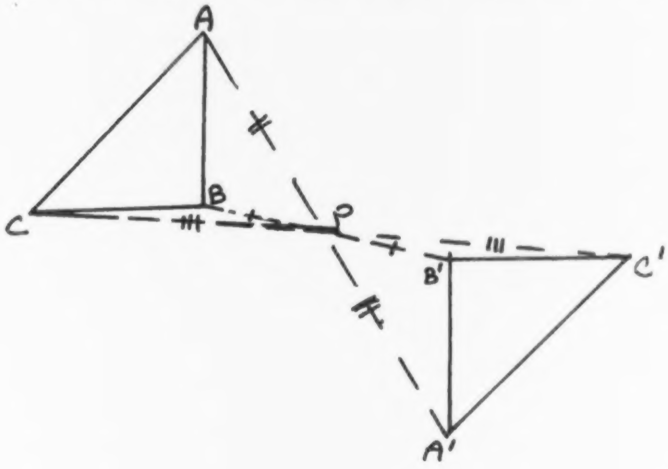
DEVELOPMENT:

1. Using the motivation problem, review the definition of line reflection and ask: "What is true of each point and its image in a line reflection with respect to the distance between the points and the line of reflection?" Elicit: The distance between any point and the line of reflection is equal to the distance between the point's image and the line of reflection. (See diagram.)



2. Given triangle ABC (see diagram) and point P, reflect point A through point P. Ask, "What is true of $\overline{PA}$ and $\overline{PA}'$ (A' is the image of A)? Elicit: $\overline{PA} \cong \overline{PA}'$. Likewise demonstrate the reflection of points B and C

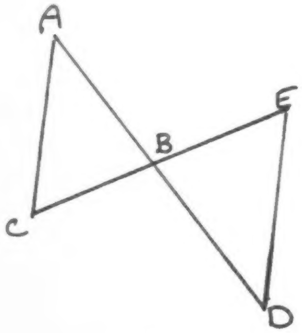
through point P. Ask, "What is true of $\overline{PB}$ and $\overline{PB}'$ and $\overline{PC}$ and $\overline{PC}'$ (B' and C' are the images of B and C respectively)? Elicit: $\overline{PB} \cong \overline{PB}'$ and $\overline{PC} \cong \overline{PC}'$.



3. Define a reflection in a point as a transformation of the plane such that:
   a) The image of fixed point P is P.
   b) For other points, the image of K is K' where P is the midpoint of KK'.
4. Do Application #1.
5. Illustrate the notation $R_p(A) = A'$ means the image of point A under a reflection in point P is A'.
6. Show that the letter O has a point (its center) such that any point reflected through this point (the center) will have its image on the circle.
7. Define point symmetry as occuring in a figure when the figure is its own image under a reflection in a point.
8 Do Application #2.
9. Do Application #3.

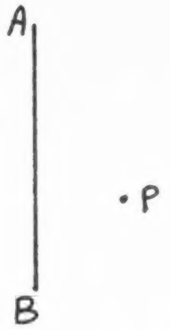
APPLICATIONS:

1. Refer to the diagram at the right. The reflection of △ABC through point B is △DBE. What is the image of A under the point reflection?

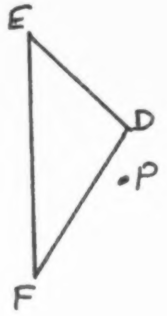


2. a) Print capital letters of the alphabet that have point symmetry.
   b) Tell which of the following words has point symmetry:
   1) SIS 2) TOOT 3) MOM 4) SWIMS

3. a) Refer to the diagram in Application #1:
   1) $R_B(C) = ?$ 2) $R_B(<CAB) = ?$
   3) $R_B(\overline{CA}) = ?$
   b) Copy the given figure and point P. Sketch the image of the figure under a reflection in point P.
   1)



   2)

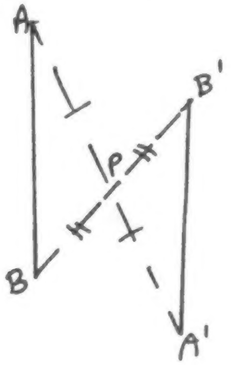


SOLUTIONS TO APPLICATION PROBLEMS:

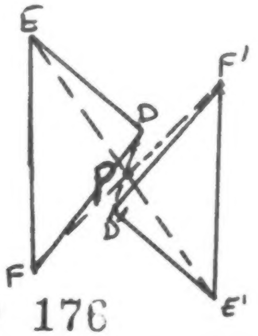
1. D

2. a) H, I, N, O, S, X, Z
   b) SIS, SWIMS

3. a) 1) E 2) <EDB 3) $\overline{ED}$
   b) 1)



   2)



SUMMARY:

1. What is meant by point reflection?
2. What is meant by point symmetry?
3. How can we determine whether a shape has point symmetry?

LESSON: 37

AIM: What is a translation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and illustrate a translation.
2. Use appropriate notation to describe a translation.
3. Find and graph the image of a given geometric figure in the coordinate plane under a translation.
4. Find a translation when a point and its image are given.

VOCABULARY: translation

MOTIVATION:

Using a ruler, draw a rectangular solid whose dimensions are 2" x 3" x 4".
Ask: "How are the back and front of the rectangular solid related?"
Elicit: They are congruent rectangles in different locations.

DEVELOPMENT:

1. Using the motivation problem, explain the 'shift' of the front face of the solid to the back face. This shift is another type of transformation.
2. Define a translation as a transformation of the plane that shifts every point in the plane the same distance in the same direction to its image.
3. Graph A(0,0), B(1,5), C(-5,-2) and their respective images A'(4,3), B'(5,8), C'(-1,1). (The teacher may wish to use an overhead projector to demonstrate these coordinates.)
   Ask: "What has occured to each pair of coordinates?"
   Elicit: x is increased by 4 and y is increased by 3.
   Ask: "Is there a transformation that maps ΔABC to A'B'C'?" Elicit: Yes, a translation.

4. Establish the notation: $T_{4,3}(x,y) = (x+4, y+3)$.

5. Do Application #1.

6. Ask: "What geometric relationship exists between the two triangles?"
   Elicit: They are congruent.
   Ask: "What properties are preserved under a translation?"
   Elicit: All properties. Demonstrate.

7. Do Application #2.

8. Do Application #3.

APPLICATIONS:

1. Graph $\triangle ABC$ whose vertices are A(-2,1), B(1,4) and C(2,2).
   a) On the same set of axes, graph $\triangle A'B'C'$ where $T_{5,-1}(\triangle ABC) = \triangle A'B'C'$.
   b) On the same set of axes, graph A"B"C", where $T_{-3,5}(\triangle A'B'C') = \triangle A''B''C''$.
   c) Name a single transformation that is equivalent to $T_{5,-1}$ followed by $T_{-3,5}$.

2. Show that the properties of distance and midpoint are preserved in Application #1.

3. A translation maps A(0,2) onto A'(2,0). Find the coordinates of B', the image of B(1,1), under the same translation.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) A'(3,0), B'(6,3) C'(7,1)
   b) A"(0,5), B"(3,8), C"(4,6)
   c) $T_{2,4}$

2. Use the distance formula to show that $\overline{AB} \cong \overline{A'B'} \cong \overline{A''B''}$, etc. Use the midpoint formula to show that the midpoint of $\overline{AB}$ and the midpoint of $\overline{A'B'}$ are images of each other, etc.

3. B'(3,-1)

SUMMARY:

1. What is meant by a translation?
2. What properties are preserved under a translation?

LESSON: 38

AIM: What is a rotation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and illustrate a rotation about a center.
2. Use appropriate notation to indicate a rotation clockwise or counterclockwise about a point x degrees.
3. Find and draw images of figures in the coordinate plane after a rotation about the origin of ±90° or ±180°.
4. Define and illustrate rotation symmetry.
5. Define and illustrate half-turn.
6. Determine whether a given figure has rotational symmetry.

VOCABULARY: rotation, half-turn, rotational symmetry

MOTIVATION:

Draw a circle with an indicated point as the center and a point labelled P on the circle.
This is a picture of a car's steering wheel with a spot at point P. Ask, "When you make a left-hand turn, what happens to point P?" Elicit: Point P moves along the wheel in a counterclockwise motion. Ask, "When you make a right-hand turn, what happens to point P?" Elicit: Point P moves along the wheel in a clockwise motion. Ask, "Does any point on the circle not move?" Elicit: No. Ask, "Does the center move?" Elicit: No.

DEVELOPMENT:

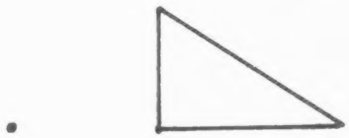
1. Using the motivation problem, ask: "Does this example define a transformation?" Elicit: Yes.
   Define a rotation as a transformation of the plane about a point, called the center of rotation and through an angle of θ degrees. The image of the fixed point P is P

and for other points the image of K is K' where $m<KPK' = \theta^\circ$ and $\overline{PK} = \overline{PK}'$.

2. Review: An angle rotated in a counterclockwise direction is defined as a positive angle and an angle rotated in a clockwise direction is defined as a negative angle.
3. Do Application #1.
4. Establish the notation for rotation. $R_{p,N^\circ}(A) = A'$ means the image of A rotated N° about point p is A'. Note that since a rotation of 180° about a point P is equivalent to a reflection in point P, we abbreviate $R_{P,180^\circ}$ as R_p. If no point is indicated, it is assumed that the rotation is taken about the origin. Therefore R_{40° means a rotation of 40° rotation about the origin.
5. Ask: "What properties are preserved under rotations?" Elicit: All. Demonstrate.
6. Draw a square and rotate it 90°. Ask: "What is true of the square when rotated 90°?" Elicit: The figure is the same as the original. Ask: "If we draw an equilateral triangle, is it possible to rotate it a certain number of degrees and produce an image which is the same as the original figure?" Elicit: Yes. Rotate it 120°.
7. Define rotational symmetry as a figure being its own image under a rotation of an angle greater than 0° but less than 360°.
   Ask: "What other figures have rotational symmetry?" Elicit: Regular polygons, the letter H, the letter X, etc. Note that if a figure has point symmetry that means it is its own image under a half-turn or a 180° rotation.
8. Do Application #2.
9. Do Application #3.

APPLICATIONS:

1. Rotate triangle ABC through an angle of 90° about point Q.

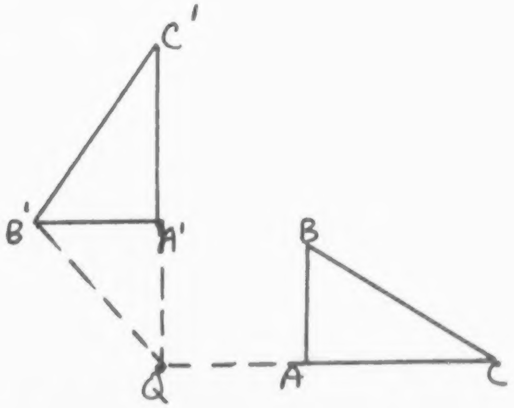


2. Find the image of $\triangle$DEF under a half-turn in the origin if the coordinates of the vertices are A(3,1), B(5,-2), and C(6,7).

3. Complete:
   a) $R_{o,180^\circ}(x,y) =$
   b) $R_{o,90^\circ}(x,y) =$
   c) Find the image of $\triangle$ ABC under a rotation of -90° about the origin if the coordinates of the vertices of $\triangle$ ABC are A(1,3), B(5,1), and C(1,1).

SOLUTIONS TO APPLICATION PROBLEMS:

1.



2. A'(-3,-1), B'(-5,2), C'(-6,-7)

3. a) (-x,-y)
   b) (-y,x)
   c) A'(3,1), B'(1,-5), C'(1,-1)

SUMMARY:

1. What is meant by a rotation about a point?
2. What is meant by rotational symmetry?
3. What properties are preserved under a rotation?

LESSON: 39

AIM: What is a dilation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and illustrate a dilation.
2. Use the appropriate notation for a dilation.
3. Find and graph the image of a figure in the coordinate plane under a dilation, D_N, with $-3 \leq N \leq 3$, $N \neq 0$

VOCABULARY: dilation, constant of dilation (scale factor)

MOTIVATION:

In photography we sometimes enlarge a print. Ask, "Would the print and its enlargement represent an example of a transformation. Elicit: Yes. Ask, "Why?" Elicit: It meets the definition of a transformation.

DEVELOPMENT:

1. Using the motivation problem, ask: "What is the relationship between the print and its enlargement?" Elicit: They are similar.
2. (The teacher may wish to use an overhead projector here.) Given $\triangle$ ABC whose coordinates are A(0,0), B(3,4) and C(2,5) and $\triangle$A'B'C' which is the image of $\triangle$ ABC under a transformation of the plane and whose coordinates are A'(0,0), B'(6,8) and C'(4,10). Ask: "What is the difference between this transformation and the others we have studied?" Elicit: These triangles are similar, while the others were congruent. Establish the specific relationship that the length of each side of $\triangle$ A'B'C' is twice the length of the corresponding side of $\triangle$ ABC. Note that this transformation is called a dilation.

3. Define a dilation of k, where $k>0$, the constant of dilation, is a transformation of the plane such that:
   a) The image of 0, the center of dilation is 0.
   b) For other points, the image of P is P' where $\overline{OP}$ and $\overline{OP}'$ name the same ray and $\overline{OP}' = k \cdot (\overline{OP})$. Refer to part 2 and indicate that the constant of dilation (scale factor) is 2.

4. Establish the notation $D_k(x,y) = (kx,ky)$ means the image of point (x,y) under a dilation of k is (kx,ky).

5. Do Application #1.

6. Do Application #2.

7. Ask, "What properties are preserved under a dilation?" Elicit: All, except distance. Demonstrate.

8. Ask, "What is meant by D_{-2}?" Elicit: $D_{-2}(x,y) = (-2x,-2y)$ which is equivalent to a dilation of 2 followed by a point reflection.

9. Do Application #3.

APPLICATIONS:

1. Find the image of △ABC under D_3 and $D_{1/2}$ if the vertices of the triangle are A(0,4), B(4,2) and C(2,-2).

2. Suggest a rule for each of the following:
   a) $D_3(x,y) =$
   b) $D_{1/2}(x,y)$
   c) $D_M(x,y)$

3. Which transformation can produce an image triangle with an area NOT equal to the area of the original triangle?

SOLUTIONS TO APPLICATION PROBLEMS:

1. Under a D_3: A'(0,12), B'(12,6), C'(6,-6)

   Under a $D_{1/2}$: A'(0,2), B'(2,1), C'(1,-1)

2. a) $(3x,3y)$
   b) $(\frac{1}{2}x, \frac{1}{2}y)$
   c) (Mx,My)

3. dilation

SUMMARY:

1. What is meant by a dilation?

2. What property is not preserved under a dilation?

LESSONS: 40, 41

AIM: What are the parts of a circle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define circle, radius, diameter, center, chord, secant, tangent, central angle, arc, semicircle, minor arc, major arc, congruent arcs, congruent circles.
2. State and apply the postulate: In the same or ≅ circles, all radii are ≅.
3. State and apply the postulate that the central angle is equal to the measure of its intercepted arc.
4. Distinguish between arc degrees and arc length.
5. Solve algebraic and numeric examples with central angles.

VOCABULARY: circle, radius, diameter, center, chord, secant, tangent, central angle, arc, semicircle, minor arc, major arc, congruent arcs, congruent circles

MOTIVATION:

Describe the locus of points five units from a given point.
Elicit: The locus is a circle with a radius of 5 units. The given point is the center of the circle.

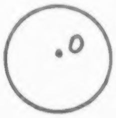
DEVELOPMENT:

1. Elicit the definition of a circle.

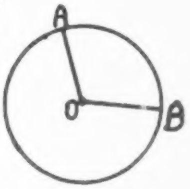
   Ask, "What is the name of the "fixed point" in our definition?"

   Definition: A circle is a set of points in a plane such that the points are equidistant from a fixed point called the center of the circle. Inform the students that we usually identify a circle by naming its center.

   This is circle O, written in symbols as ⊙O.



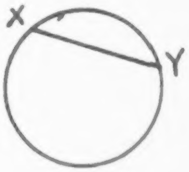
2. Draw $\overline{OA}$ from the center to the circle O. (see diagram below)



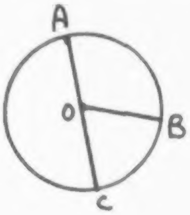
Define radius: A radius of a circle (plural = radii) is a line segment from the center of the circle to any point of the circle. Draw $\overline{OB}$.

Ask: What is true about $\overline{OA}$ and $\overline{OB}$? Elicit that they are equal in length. Why? Since every point of the circle is equidistant from its center $\overline{OA} \cong \overline{OB}$. State the postulate: In the same or $\cong$ circles, all radii are $\cong$.

3. Draw a chord $\overline{XY}$. (see diagram below)
Define chord: A chord of a circle is a line segment whose endpoints are points of the circle.



4. Extend $\overline{OA}$ to meet circle O at C. $\overline{AC}$ is called diameter. (see diagram below)
Define diameter: A diameter of a circle is a chord that passes through the center. What is the relationship between a diameter and a radius of a circle?



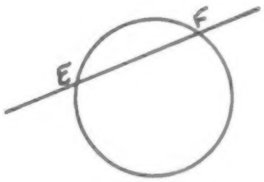
$\overline{OA} \cong \overline{OC}$ All radii of same circle are congruent

$\overline{AC} = \overline{OA} + \overline{OC}$

Therefore $d = 2r$ if the length of a radius of a circle is r and the length of a diameter of that circle is d.

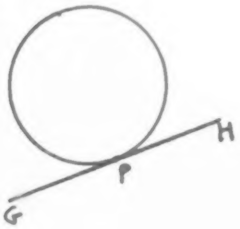
Ask, "Is a diameter always a chord?" (Yes) Explain.
"Is a chord always a diameter?" (No) Explain.

5. Definition: A secant is a line which intersects a circle in two points.

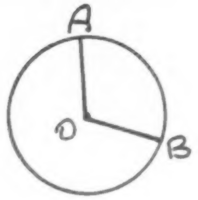


$\overleftrightarrow{EF}$ is a secant

6. Definition: A tangent to a circle is a line which intersects a circle at one and only one point. The point at which the tangent intersects the circle is called the "point of tangency" or "point of contact".



$\overleftrightarrow{GH}$ is a tangent to circle O. Point P is the point of tangency or point of contact.



7. Consider angle AOB

Where is its vertex?

This is called a central angle.

Elicit the definition: A central angle of a circle is an angle whose vertex is the center of the circle and whose sides are radii. (Recall that an angle is the union of two rays having a common endpoint (vertex).)

In each case, find the measure of angle AOB.

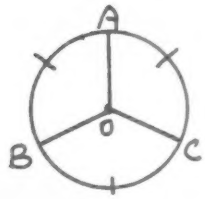


FIGURE 1

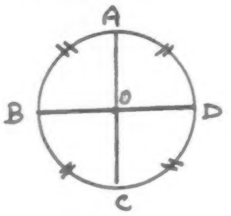


FIGURE 2

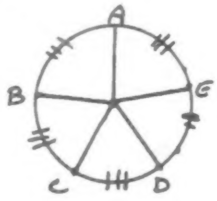


FIGURE 3

Elicit that ∡ AOB is a central angle.

Since the circle is divided into equal parts, each part or ARC has the same measure.

Explain to class:

An arc is a part of a circle with two endpoints on the circle. The whole circle measures 360°.

In figure 1, m $\widehat{AB}$ = 120

In figure 2, m $\widehat{AB}$ = 90

In figure 3, m $\widehat{AB}$ = 72

A central angle is equal in measure to its intercepted arc.

An intercepted arc is an arc (part of a circle) cut off by a chord or an angle.

Do Application 1.

Do Application 2.

8. Explain to class:

Types of Arcs:

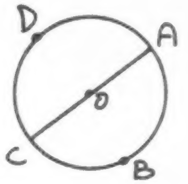
An arc of a circle is any part of the circle.

A semicircle is an arc that is half a circle.

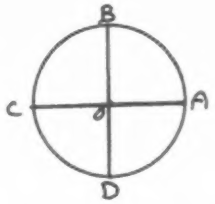
A quadrant is an arc that is one quarter of a circle.

A minor arc is an arc that is less than a semicircle.

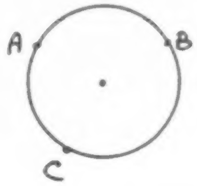
A major arc is an arc that is more than a semicircle.



$\widehat{AC}$ is a semicircle
$\widehat{ADC}$ is a semicircle
$\widehat{ABC}$ is a semicircle



$\widehat{AB}$ is a quadrant
$\widehat{AD}$ is a quadrant
$\widehat{CD}$ is a quadrant
$\widehat{BC}$ is a quadrant



Minor Arc = $\widehat{AB}$
Major Arc = $\widehat{ACB}$

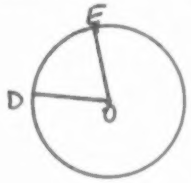
Do Application 3.

Do Application 4.

9. Explain to class:

The degree measure of an arc is equal to the measure of the central angle that intercepts the arc.

In circle O if $m\angle DOE = 60$, then the degree measure of arc DE is also 60°, written as $m\,\widehat{DE} = 60$.



Remember the degree measure of an arc is NOT the length of an arc. For example, if the circumference of circle O is 12 cm, then the length of DE is

$$\frac{60}{360} \cdot (12 \text{ cm}) = \frac{720}{360} \text{ cm} = 2 \text{ cm}$$

Therefore: The degree measure of arc DE is 60°.
The length of the arc DE is 2 cm.

LESSON: 63

AIM: How do we measure an angle using radians?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a radian.
2. Find the number of radians in an angle when given the number of degrees.
3. Find the number of degrees in an angle when given the number of radians.
4. Find the length of an arc in a circle when given the central angle measured in radians and the length of the radius.

VOCABULARY: radian

MOTIVATION:

Which is greater 180° or 3?

Students should realize that they can not compare the two quantities because 3 is a real number and 180° is a measure of rotation. In this lesson, students will learn a new unit for measuring rotation. Just as length may be measured in different ways, rotation is measured using two different units. The advantage of the new unit of measure is that it is also a real number and may be compared in size to other real numbers. This will be most useful later in the unit when students learn to graph trigonometric functions.

DEVELOPMENT:

1. Review the method for finding the length of an arc, using the formula $\frac{n}{360^\circ} = \frac{s}{2\pi r}$. n represents the angle measured in degrees and s is the length of the arc. Have the class use the formula to find n in the following two problems:
    1) $r = 6$ and $s = 6$
    2) $r = 12$ and $s = 12$

Students should get $n = \frac{180°}{\pi}$ for both problems. Furthermore, show that if r = s then $n = \frac{180°}{\pi}$ for all circles. Now define radian - is the measure of a central angle that intercepts an arc equal in length to the radius. 1 radian = $\frac{180°}{\pi}$ 57°

2. We can convert the number of radians to degrees in the following way: $\theta = \frac{2\pi}{3}$ then $\theta = \frac{2\pi}{3} \cdot \frac{180°}{\pi} = 120°$

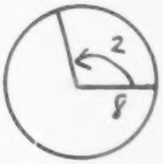
   To convert from degrees to radian, reverse the process.

   $\theta = 80°$ $\theta = 80° \cdot \frac{\pi}{180} = \frac{4\pi}{9}$

   Radians do not require a symbol because they are real numbers. π radians = 180°

3. Our original problem was to compare 3 to 180°. Since 180° = π then we can compare π to 3.

4. In the circle the measure of the central angle is 2 radians and the radius = 8 inches. We may find the



length of the arc by multiplying the radius by 2 because each radian intercepts an arc of 8 inches; therefore, s = 16 inches.

We generalize in the following way: If s represents the length of the arc, r represents the radius and θ represents the central angle measured in radians then $s = R\theta$ is a formula for measuring the length of the arc. The major difference between this formula for length of arc and the original formula is that the central angle is measured in radians.

APPLICATIONS:

1. Convert each to degrees.
   a) $\frac{2\pi}{5}$ b) $\frac{3\pi}{2}$ c) $\frac{-\pi}{4}$ d) $\frac{5\pi}{2}$

2. Convert each to radians.
   a) 70° b) 150° c) -200° d) 400°

3. If a central angle of a circle equals 3 radians and the radius in 7 inches, then what is the length of the arc?

4. The length of an arc of a circle is 8 inches and the radius of the circle is 12 inches. Find the measure of the central angle in radians.

5. Find the value of each: a) sin $\frac{\pi}{3}$ b) cos $\frac{\pi}{4}$ c) tan $\frac{\pi}{6}$ without using trig table.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\frac{2\pi}{5} \cdot \frac{180}{\pi} = 72°$ b) $\frac{3\pi}{2} \cdot \frac{180}{\pi} = 270°$
   c) $\frac{-\pi}{4} \cdot \frac{180}{\pi} = -45°$ d) $\frac{5\pi}{2} \cdot \frac{180}{\pi} = 450°$

2. a) $90° \cdot \frac{\pi}{180°} = \frac{\pi}{2}$ b) $150°(\frac{\pi}{180°}) = \frac{5\pi}{6}$
   c) $-200° \cdot \frac{\pi}{180} = \frac{-10\pi}{9}$ d) $400° \cdot \frac{\pi}{180}) = \frac{20\pi}{9}$

3. $s = r\theta$
   $s = 7(3)$
   $s = 21$ inches

4. $s = r\theta$
   $8 = 12\theta$
   $\theta = \frac{2}{3}$

5. a) $\sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$
   b) $\cos\frac{\pi}{4} = \cos 45° = \frac{\sqrt{2}}{2}$
   c) $\tan\frac{\pi}{6} = \tan 30° = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

SUMMARY:

1. What is a radian?

2. What is the exact number of degrees in a radian? What is the approximate number of degrees in a radian?

3. What is the advantage of using radians over using degrees to measure rotation?

4. What is π radians equal to in a) degrees b) length?

5. What is the formula for measuring arc length using a) radians b) degrees?

LESSON: 64

AIM: How do we define the sine, cosine and tangent of an angle using standard position?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the sine, cosine and tangent of an angle using the coordinates of the point of intersection of the unit circle and a point on the terminal side of the angle.
2. Find the sine, cosine and tangent of an angle when given the coordinates of a point on the terminal side of the angle.
3. Determine whether the sine, cosine and tangent of an angle is either positive or negative when given the measure of the angle.

VOCABULARY: unit circle

MOTIVATION:

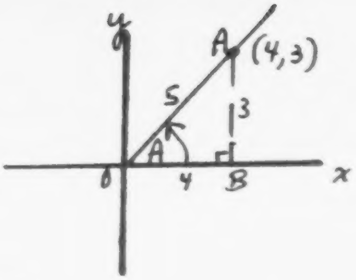
A point on the terminal side of angle A has coordinates (-4, 3).

a) Draw angle A in standard position.
b) Find the distance of point A from the origin.
c) Find sin A.

Students will be able to do part a and part b but should not be able to do part c because the sine of an angle is defined in terms of a right triangle and an angle in the second quadrant can not be a part of a right triangle.
Explain that the aim of the lesson is to define the trigonometric functions so that we can evaluate them for any size angle.

DEVELOPMENT:

1. Begin with the following problem: A point on the terminal side of angle A has coordinates (4, 3). Find sin A. Do the problem with the class in the following way:



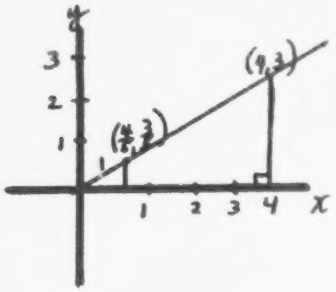
a) Draw the angle in standard position.

b) Find the distance of point A from the origin.
$r = \sqrt{4^2+3^2} = 5$

c) Draw a $\perp$ from point A to x-axis.

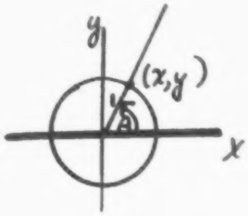
d) Using right triangle AOB find sin A. Sin A = $\frac{3}{5}$

Point out that we were able to find the sin A using only a point on the terminal side of A. If we now divide both coordinates of A by 5 we now have a new triangle whose legs are $\frac{3}{5}$ and $\frac{4}{5}$ and whose hypotenuse is 1 as shown in the diagram below.



Notice that the new triangle is similar to the original triangle and that the sin A is equal to the y-coordinate and cos A is equal to the x-coordinate. Tan A is equal to $\frac{y}{x}$

2. We are now ready to define sin A, cos A and tan A using the standard position of the angle. Draw angle A in standard position, as in the diagram. Draw a unit circle with the origin as the center.



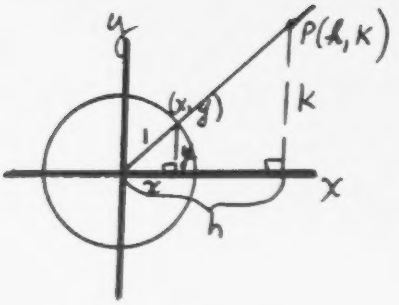
Angle A intersects the unit circle in the point (x, y). The distance of the point from the origin is 1, so that $x^2+y^2=1$.

The new definition is: $\sin A = y$

$\cos A = x$

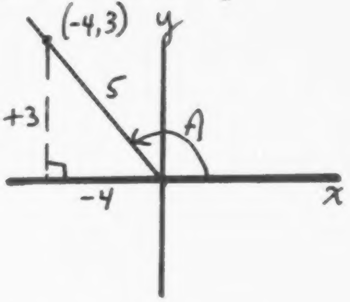
$\tan A = \frac{y}{x}$

This definition is equivalent to the original because of the following explanation:



a) Given angle A is acute, draw angle A in standard position.
b) Take point P on terminal side with coordinates (h, k) and distance from origin r.
c) sin A = $\frac{k}{r}$ cos A = $\frac{h}{r}$ and tan A = $\frac{k}{h}$
d) Draw unit circle. It will intersect the unit circle at (x, y) with distance from origin = 1.
e) Since the 2 triangles are similar x = $\frac{h}{r}$ and y = $\frac{k}{r}$ and, therefore, sin A = y and cos A = x and tan A = $\frac{y}{x}$

3. The advantage of the new definition can be found by referring back to the motivation problem and drawing the diagram. Divide each coordinate by 5 and we get coordinates $(\frac{3}{5}, \frac{-4}{5})$. This will be coordinates on unit circle.
cos A = $\frac{-4}{5}$, sin A = $\frac{3}{5}$ tan A = $\frac{-3}{4}$



By using the new definition we were able to find the sine, cosine and tangent of an angle which is not in a right triangle.

4. From our answers to the above problem we see that sin A and Tan A are negative. We will now explore all the different cases to find the signs of the trigonometric functions in each quadrant. To do this have the students fill in the tables below:

	I	II	III	IV
x	+	-	-	+
y	+	+	-	-
x	+	-	+	-

	I	II	III	IV
sin A = y	+	+	-	-
cos A = x	+	-	-	+
tan A = $\frac{y}{x}$	+	-	+	-

Summarize the results in the second table by using the mnemonic, "All Students Take Calculus".

Students Sine +	All +
Take Tangent +	Calculus Cosine +

In I all functions are positive.
In II only sine is positive.
In III only tangent is positive.
In IV only cosine is positive.

If a function is not positive in a quadrant, then it is negative in that quadrant. All functions are positive in two quadrants and negative in two quadrants.

APPLICATIONS:

1. Point A is on the terminal side of angle θ.
   Find sin θ, cos θ and tangent θ in each problem.
   a) A(-5, 12) b) (0, -2) c) (3, -1) d) $(\frac{3}{5}, \frac{-4}{5})$

2. In each problem find the quadrant in which angle θ lies.
   a) $\sin\theta > 0$ and $\cos\theta < 0$
   b) $\tan\theta < 0$ and $\sin\theta < 0$
   c) $\cos\theta > 0$ and $\tan\theta > 0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) x = -5, y = 12, r = 13, Point on unit circle is
   $(\frac{-5}{13}, \frac{12}{13})$ $\sin\theta = \frac{12}{13}$ $\cos\theta = \frac{-5}{13}$ $\tan\theta = \frac{12}{-5}$
   b) x = 0, y = -2, r = 2, Point on unit circle is (0, -1)
   $\sin\theta = -1$ $\cos\theta = 0$ $\tan\theta = \frac{-1}{0}$ = Undefined
   c) x = 3, y = -1, r = 2, Point on unit circle is $(\frac{3}{2}, \frac{-1}{2})$
   $\sin\theta = \frac{-1}{2}$ $\cos\theta = \frac{3}{2}$ $\tan\theta = \frac{-1}{3}$
   d) $x = \frac{3}{5}$, $y = \frac{-4}{5}$, r = 1, $(\frac{3}{5}, \frac{-4}{5})$
   $\sin\theta = \frac{-4}{5}$ $\cos\theta = \frac{3}{5}$ $\tan\theta = \frac{-4}{3}$

2. a) II
   b) IV
   c) I

SUMMARY:

1. What are the definitions of the trigonometric function using a) a right triangle b) a unit circle?
2. What is the major advantage of defining the trigonometric functions using a unit circle?
3. When given a point on the terminal side of an angle, how do we find the trigonometric function of that angle?
4. What is a mnemonic device for determining the quadrants in which the trigonometric functions are positive?

LESSON: 65

AIM: How do we find the other two trigonometric function values of an angle when given one trigonometric function value?

How do we find the trigonometric function values of the quadrantal angles.

PERFORMANCE OBJECTIVES: The student will be able to...

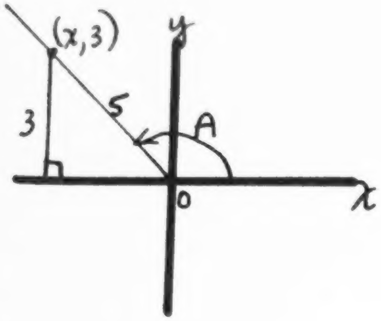
1. Find the other two trigonometric function values of an angle when given one of the trigonometric function value.
2. Find the trigonometric function values of the quadrantal angles.

MOTIVATION:

If $\sin A = \frac{3}{5}$ and $\cos A < 0$, then what is the value of cos A?

Students should realize that angle A is in the second quadrant but should be unable to go further in this problem.

DEVELOPMENT:



1. Have the class draw the angle in standard position as in the diagram. They should be able to state that the y-value of a point on the terminal side is $\frac{3}{5}$ when the distance from the origin is 1. Instead of using the point on the unit circle have the class locate the point on the terminal side whose ordinate is 3 and whose distance from the origin is 5 and then draw a $\perp$ to the x-axis. The abscissa of the point is unknown but can be found by using formula $x^2+y^2 = r^2$. Students should find that $x = \pm 4$. Choose -4 because the abscissa of a point in the second quadrant is negative. Therefore $\cos A = \frac{-4}{5}$.

Point out that we would get the same answer if we use the point on the unit circle because the triangle formed by drawing a $\perp$ from the point on the unit circle to the x-axis is similar to the triangle we used.

2. Present the problem, if $\sin A = 1$, find a) $\cos A$ b) the measure of angle A.
   Students should discover that $A=90°$ or $\frac{\pi}{2}$ and that the abscissa is zero and, therefore, $\cos A = 0$. They should also conclude that $\tan A$ is undefined.

3. Carry this problem one step further by asking the class how they could find the trigonometric functions of another quadrantal angle such as 180° or π. For 180° students can use the point $(-1, 0)$ and find $\sin 180° = 0$, $\cos 180° = -1$ and $\tan A = 0$.

APPLICATIONS:

1. If $\cos A = \frac{-5}{13}$ and $\sin A < 0$, then find $\tan A$.

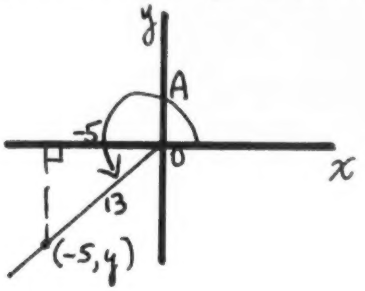
2. If $\tan \theta = \frac{-2}{3}$ and $\cos \theta > 0$, then find $\sin^2\theta + \cos^2\theta$.

3. Find each of the following.
   a) $\sin 270°$ b) $\cos \frac{\pi}{2}$ c) $\tan \frac{3\pi}{2}$ d) $\cos 2\pi$

4. If $A = \pi$, then find the value of
   $\sin \frac{1}{2}A - \cos^2 \frac{\pi}{6}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. Angle A is in quadrant III



$13^2 = (-5)^2+y^2$

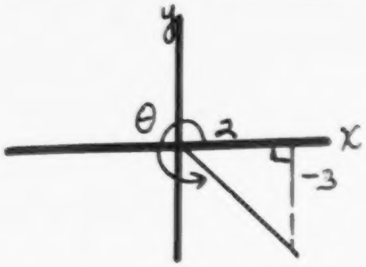
$y^2 = 144$

$y = \pm 12$

Let $y = -12$ because y is negative in quadrant III

$\tan A = \frac{-12}{-5} = \frac{12}{5}$

2. Angle θ is in quadrant IV



$r^2 = x^2+y^2$

$r^2 = 2^2+(-3)^2$

$r = \sqrt{13}$; r is always positive

$\sin\theta = \frac{-3}{\sqrt{13}}$; $\cos\theta = \frac{2}{\sqrt{13}}$

$$\sin^2\theta + \cos^2\theta = \left(\frac{-3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{9}{13} + \frac{4}{13} = 1$$

3. a) $\sin 270° = -1$; Use pt (0, -1)

   b) $\cos \frac{\pi}{2} = 0$; Use pt (0, 1)

   c) $\tan \frac{3\pi}{2}$ Undefined; Use pt (0, -1)

   d) $\cos 2\pi = 1$; Use pt (1, 0)

4. $\sin \frac{1}{2}\pi - \cos^2 \frac{\pi}{6}$

   $1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$

SUMMARY:

1. When given one trigonometric function and the quadrant of an angle, how can we find the other trigonometric functions of the angle? Elicit the following steps:
   a) Draw the angle in standard position.
   b) Draw a ⊥ to the x-axis and put the known values on the sides of the triangle formed.
   c) Find the third side of the triangle using the formula $x^2+y^2=r^2$. Remember that r is always positive and the sign on x and y is determined by the quadrant of the angle.
   d) Find the unknown trigonometric functions using the value found in part c.
2. What are the points of intersection of the unit circle and the coordinate axes?
   How are those points used in finding the trigonometric functions of the quadrantal angles?
3. When is the tangent of an angle undefined?

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